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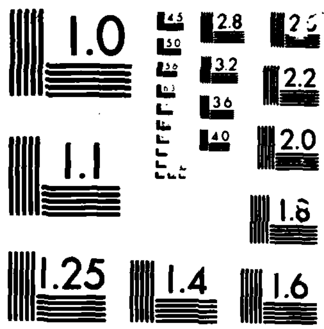

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A classification scheme is presented for traveling wave solutions of reaction diffusion systems of the form $x_t = x_{\alpha\alpha} + \nabla V(x)$ where $t, \alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $V: \mathbb{R}^n \to \mathbb{R}$. The important assumptions on V are that $\lim_{\|x\|\to\infty} V(x) = -\infty$, that the set $\{x \mid V(x) > -Q\}$ is convex for Q sufficiently

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Classification of Traveling Wave Solutions of Reaction-Diffusion Systems

by

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December 1985

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Abstract

A classification scheme is presented for traveling wave solutions of reaction diffusion systems of the form $x_t = x_{\alpha\alpha} + \nabla V(x)$ where $t, \alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $V: \mathbb{R}^n \to \mathbb{R}$ The important assumptions on V are that $\lim_{\|x\| \to \infty} V(x) = -\infty$, that the set $\{x \mid V(x) > -Q\}$ is convex for Q sufficiently large, that V has a finite number of critical points, and that if M_1 and M_2 are critical points of V then $V(M_1) \neq V(M_2)$.

The primary tools used are the Conley index and connection matrix. The classifications are given via paths in graphs whose vertices and edges are connection matrices. These results are then used to prove the existence of an infinite number of traveling wave solutions for a specific example.

Introduction

Many chemical and biological systems have been modeled by systems of reaction-diffusion equations. A simple system of this type is

$$X_\tau = X_{\alpha\alpha} + \nabla V(x) \tag{0.1}$$

where $\alpha,\tau \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $V:\mathbb{R}^n \to \mathbb{R}$. A typical problem is to prove the existence of a special type of solution called a traveling wave. A traveling wave solution to (0.1) is a non constant bounded solution of the form $x(\alpha,\tau) = x(t)$ where $t = \alpha + \theta\tau$. This solution must satisfy the non-linear system of O.D.E.'s

$$\begin{aligned} \dot{x}_i &= y_i \\ \dot{y}_i &= \theta y_i - D_i V(x) \qquad \cdot = \frac{d}{dt} \end{aligned} \tag{0.2}$$

with boundary conditions that $\lim_{t\to\infty} (x(t),y(t)) = (M_i,0)$ and $\lim_{t\to-\infty} (x(t),y(t)) = (M_j,0)$ where M_i and M_j are critical points of the potential function V. Such a solution is called an $M_j \to M_i$ connection and is said to occur at the wave speed θ. This paper presents a method for classifying the possible traveling wave solutions to (0.1) which relies only on information about the critical points of V. The necessary assumptions on V are given in Section 2.1.

The primary tools used for the classification are the Conley index and connection matrix. The Conley index, $h(\cdot)$, associates to each critical point

of V a pointed k-sphere denoted Σ^k. We call a connection $M_i \to M_j$ a degree ℓ connection if $h(M_i) \sim \Sigma^k$ and $h(M_j) \sim \Sigma^{k+\ell}$. The connection matrix gives information about degree -1 connections and, in some sense, is stable under perturbation. However, the traveling wave solutions of (0.1) which are of primary interest are degree 0 connections and hence correspond to non-transverse heteroclinic solutions to (0.2). As such they will in general only occur for a discrete set of θ values. What will be shown is that the connection matrix changes precisely at the wave speeds for which degree 0 connections occur. This in turn can be used to determine the possible degree 0 and degree -1 connections at various wave speeds for a fixed potential function V.

This paper is divided into four sections. The first, consists of a brief review of the Conley index and a short discussion of the connection matrix. Sources for some of the material in 1.1 are Conley [1], Conley-Zehnder [2] and Salamon [8]. For a discussion of the index filtration and connection matrix the reader is refered to Franzosa [3], [4]. The second section presents the simple analytic results for (0.2) and two other related systems which will be studied. In addition, the results are translated into the language of Conley. The third section applies the connection matrix techniques to the three systems of interest and integrates these various results to develop the classification scheme. Also, the question of how to interpret the results is discussed. The classification is given in terms of elementary transition graphs. This was motivated by the work of Terman [9] who studied (0.1) where $x \in \mathbb{R}$ His results are more geometrical in nature and limited to the 1-dimensional problem but many of them can be reproduced via the elementary transition

graphs. The fourth and final part consists of two examples. In 4.1, several elementary transition graphs are constructed. In 4.2, one of these graphs is used to conclude the existence of an infinite number of wave speeds for which certain degree 0 connections occur.

This problem was suggested to me by Charles Conley and many of the results were motivated by many enjoyable conversations with him. I would also like to thank Jim Reineck and Liz Mansfield for sharing their ideas with me.

1.1 Conley Index

Let the continuous map $\phi: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ denoted by $\phi(z,t) = z \cdot t$ be a flow, i.e. $z \cdot 0 = z$ and $(z \cdot t) \cdot s = z \cdot (t+s)$ for all $z \in \mathbb{R}^n$, $s,t \in \mathbb{R}$ Given $A \subset \mathbb{R}^n$, its ω and ω^* limit sets are given by

$$\omega(A) = \bigcap_{t>0} \mathrm{cl}(A \cdot [t,\infty))$$

$$\omega^*(A) = \bigcap_{t>0} \mathrm{cl}(A \cdot (-\infty,t])$$

where $\mathrm{cl}(X)$ is the closure of the set X.

Definition 1.1 A Partially ordered set is a pair $(P,>)$ *consisting of a set,* P, along with a partial order relation, $>$, satisfying:

1) $\pi > \pi$ never holds for $\pi \in P$

2) If $\pi > \pi'$ and $\pi' > \pi''$ then $\pi > \pi''$.

In the applications it is often clear what order relation is being assumed and hence *one often writes* P *instead of* $(P,>)$.

An interval in P is a subset, $I \subset P$, for which $\pi,\pi' \in I$ and $\pi > \pi'' > \pi'$ implies that $\pi'' \in I$. An attracting interval in P is an interval, I, such that $\pi \in I$ and $\pi > \pi'$ implies $\pi' \in I$. Now let I and J be disjoint intervals, then (I,J) is an adjacent pair of intervals if:

1) $J \cup I$ is an interval.

2) $\pi \in I$ and $\pi' \in J$ *implies* $\pi \not> \pi'$.

Definition 1.2. A set $S \subset \mathbb{R}^n$ is invariant if $S \cdot \mathbb{R} = S$. S is an isolated invariant set if there exists a compact neighborhood, N, of S such that S is the maximal invariant set in N. In this case, N is called an isolating neighborhood of S.

Definition 1.3. Let S_1 and S_2 be compact invariant sets. $C(S_1,S_2) \equiv \{z \mid \omega^*(z) \subset S_1$ and $\omega(z) \subset S_2\}$. $C(S_1,S_2)$ is called the connecting set from S_1 to S_2.

Definition 1.4. Let S be a compact isolated invariant set. A Morse decomposition of S is a collection, $M = \{M(\pi) \mid \pi \in (P,>)\}$, of compact invariant sets in S. Furthermore, the sets $M(\pi)$ and the partial order are related in the following manner. Let $z \in S \backslash \{\bigcup_{\pi \in P} M(\pi)\}$. Then there exists $\pi,\pi' \in P$ with $\pi' > \pi$ such that $z \in C(M(\pi'),M(\pi))$. The individual sets, $M(\pi)$, are called Morse sets.

It is important to recognize that given S there may be different collections $\{M(\pi)\}$ which give rise to a Morse decomposition. In addition, each collection may have several admissable orders. For example, let S be as in Figure 1. The possible collections of Morse sets are

$\mathcal{M}_1 = \{M(a),M(b),M(c)\}$, $\quad \mathcal{M}_2 = \{M(a) \cup M(b) \cup C(M(a),M(b)),\ M(c)\}$

$\mathcal{M}_3 = \{M(a) \cup M(c) \cup C(M(a),M(c)),\ M(b)\}$ or $\mathcal{M}_4 = \{S\}$.

Given $\mathcal{M}_1$, admissable orderings for $P = \{a,b,c\}$ are $a > b > c$; $a > c > b$; or $a > b$, $a > c$.

INSERT FIGURE 1.

Never-the-less, given the collection of Morse sets, there is a minimal ordering which is admissable. This is called flow defined partial order, $>_F$, and is obtained by setting $\pi >_F \pi'$ if and only if there exists a sequence of distinct elements of P, $\pi' = \pi_0, \ldots, \pi_n = \pi$ such that $C(M(\pi_k),M(\pi_{k-1})) \neq \emptyset$ for all $k = 1, \ldots, n$.

Let I be an interval in P. Define

$$M(I) = \{z \mid \omega(z) \subset M(\pi),\ \omega^*(z) \subset M(\pi') \quad \text{where} \quad \pi,\pi' \in I\}.$$

It is easily checked that $M(I)$ is an isolated invariant set.

Definition 1.5. Let N be compact. A subset $K \subset N$ is positively invariant in N if $z \in K$, $t \geq 0$, and $z \cdot [0,t] \subset N$ implies $z \cdot t \in K$.

Definition 1.6. Let S be an isolated invariant set. A pair (N,N_0) of compact sets is an index pair for S if:

1) $cl(N_1 \backslash N_0)$ is an isolating neighborhood of S
2) N_0 is positively invariant in N.
3) If $z \in N_1$ and $z \cdot [0,\infty) \subset N_1$, then there exists a $t \geq 0$ with $z \cdot [0,\epsilon] \subset N_1$ and $z \cdot t \in N_0$.

Given an index pair, (N_1,N_0), N_0 is called the exit set and (3) implies that any orbit which leaves N_1 in positive time has to go through N_0.

Definition 1.7. Let $\mathcal{M} = \{M(\pi) \mid \pi \in P\}$ be a Morse decomposition of the

isolated invarient set S. An <u>index-filtration</u> for M is a collection of compact sets, $N = \{N(I)\}$, indexed by the set of attracting intervals and satisfying:

1) If I is an attracting interval then $(N(I), N(\emptyset))$ is an index pair for $M(I)$.
2) If I_1 and I_2 are attracting intervals then $N(I_1) \cap N(I_2) = N(I_1 \cap I_2)$ and $N(I_1) \cup N(I_2) = N(I_1 \cup I_2)$.

Franzosa [3] showed that an index filtration for M can always be constructed. Furthermore, he showed that if (I,J) are an adjacent pair such that $I \cup J$ is an attracting interval then $(N(I \cup J), N(I))$ is an index pair for $M(J)$.

Let (N_1, N_0) be a compact pair and define an equivalence relation, $\sim$, on N_1 by

$$z \sim z \quad \text{if} \quad z \in N_1 \quad \text{and} \quad z \sim z' \quad \text{if} \quad z, z' \in N_0.$$

then $N_1/N_0 \equiv \{z \,|\, z \in N_1 \backslash N_0\} \cup \{N_0\}$. If $p\colon N_1 \to N_1/N_0$ is the obvious projection map then N_1/N_0 can be topologized by letting $U \subset N_1/N_0$ be open if and only if $p^{-1}(U)$ is open in N_1.

<u>Definition 1.8.</u> Let S be an isolated invarient set with index pair (N_1, N_0). The <u>Conley index</u>, $h(S)$, is the homotopy type of $(N_1/N_0, N_0)$.

Conley [1] proves that this index is well defined. For purposes of computation

it is much easier to use a homology or co-homology theory rather than a homotopy theory. Using Cech co-homology one has that

$$H^*(h(S)) \simeq H^*(N_1, N_0).$$

where (N_1, N_0) is any index pair for S. Using this theory one could define a degree 1 connection matrix which is a co-boundary operator. For the author, however, singular homology theory is more intuitive and leads to an easier geometric understanding of the results. In order to have $H_*(h(S)) \simeq H_*(N_1, N_0)$ for singular homology it is necessary to restrict the set of possible index pairs such that N_0 is an absolute neighborhood retract of N_1 (See Munkres [5]).

<u>Definition 1.9.</u> An index pair is <u>regular</u> if the function $\tau : N_1 \to [0,\infty]$ defined by

$$\tau(z) = \begin{cases} \sup \{t > 0 \mid z\cdot[0,t] \subset N_1 \backslash N_0\} & z \in N_1 \backslash N_0 \\ 0 & z \in N_0 \end{cases}$$

is continuous. A <u>regular index filtration</u> is an index filtration such that for every attrating interval, I, $(N(I),N(\emptyset))$ is a regular index pair.

If (N_1,N_0) is a regular index pair for S then $H_*(h(S)) \simeq H_*(N_1,N_0)$.

Let I be an interval in P. $\hat{I}$ will always denoted an interval such that $(\hat{I},I)$ is an adjacent pair and $\hat{I} \cup I$ is an attracting interval in P. In general $\hat{I}$ is not unique, however, given I, $\hat{I}$ always exists. The following

lemma will be needed in the proof of theorem 1.11.

<u>Lemma 1.10.</u> (Salamon [8]) Let (N_1,N_0) be an index pair for S. Then there exists a continuous Lyapanov function $g : N_1 \to [0,1]$ such that:

1) $g(z) = 1$ <u>iff</u> $z\cdot[0,\infty) \subset N_1$ and $\omega(z) \subset S$.

2) $g(z) = 0$ <u>iff</u> $z \in N_0$.

3) if $t > 0$, $0 < g(z) < 1$, and $z\cdot[0,t] \subset N_1$ then $g(z\cdot t) < g(z)$.

Furthermore, if $\epsilon > 0$ and $N_\epsilon(N_0) \equiv \{z \in N_1 | g(z) \leq \epsilon\}$ then (N_1,N_ϵ) is a regular index pair for S.

<u>Proposition 1.11</u>. There exists a regular index filtration such that given I, an interval in P, $(N(\hat{I} \cup I), N(\hat{I}))$ is a regular index pair for $M(I)$.

<u>Proof</u>. Let $\bar{N} = \{\bar{N}(I) | I$ an attracting interval in $P\}$ be an index filtration. Using lemma 1.10, a regular index filtration $N = \{N(I) | I$ an attracting interval in $P\}$ will be constructed. Let $K \subset P$. Define $\mathcal{B}(K) = \{\pi \in K | \text{ if } \pi' \in K \text{ then } \pi \not> \pi'\}$. Let $L_1 = \mathcal{B}(P)$, $L_k = \mathcal{B}(P\setminus \bigcup_{i=1}^{k-1} L_i)$, $\mathfrak{L}_k = \bigcup_{i=1}^{k} L_i$ and $T_k = P\setminus\mathfrak{L}_k$. Notice that T_k is an interval in P and hence $(\bar{N}(P),\bar{N}(\mathfrak{L}_k))$ is an index pair for $M(T_k)$.

By definition of an index filtration, $(\bar{N}(P), \bar{N}(\emptyset))$ is an index pair. By lemma 1.10, $(\bar{N}(P), \bar{N}_\epsilon(\emptyset))$ is a regular index pair for S. Similarly, $(\bar{N}(P),\bar{N}_\epsilon(\mathfrak{L}_1))$ is a regular index pair for $M(T_1)$. Let $N(\emptyset) = \bar{N}_\epsilon(\emptyset)$.

Let $\pi \in L_1$. Define $N(\pi) = \{z \in \bar{N}_\epsilon(\mathfrak{L}_1) | \; t \geq 0,\ z\cdot t \in \bar{N}(\pi)\} \cup \bar{N}_\epsilon(\emptyset)$. Notice that $N(\pi) \cap N(\pi') = N(\emptyset)$ if $\pi \neq \pi'$. Furthermore $N(\pi)$ is compact.

Let τ_0: $\bar{N}(P) \to [0,\infty]$ be as in definition 1.9 for the index pair $(\bar{N}(P),N(\emptyset))$. Let $I \subset L_1$ then $N(I) \equiv \bigcup_{\Pi \in I} N(\Pi)$. Given $\hat{I}$, $I \subset L_1$ define σ: $N(I \cup \hat{I}) \to [0,\infty]$ by

$$\sigma(z) = \begin{cases} \sup \{t > 0 \mid z\cdot[0,t] \quad N(I \cup \hat{I})\backslash N(\hat{I}) & \text{if} \quad z \in N(I \cup \hat{I})\backslash N(\hat{I}) \\ 0 & \text{if} \quad z \in N(\hat{I}) \end{cases}$$

Showing that σ is continuous gives that $(N(I \cup \hat{I}'), N(\hat{I}))$ is a regular index pair. There are three cases to consider.

<u>Case 1.</u> ($z_0 \in N(I \cup \hat{I})\backslash N(\hat{I})$). $N(I \cup \hat{I})\backslash N(\hat{I})$ is open in $N(I \cup \hat{I})$, hence there exists an open set U, such that $z_0 \in U \subset N(I \cup \hat{I})\backslash N(\hat{I})$. Now let $z \in U$ then $z \in N(\Pi)$ where $\Pi \in I$. The orbit of z passes through $N(\emptyset)$ without passing through $N(I)$. Thus $\sigma|_U = \tau_0|_U$. Thus σ is continuous at z_0.

<u>Case 2.</u> ($z_0 \in N(\hat{I})\backslash N(\emptyset)$). $N(I) \cap N(\hat{I}) = N(I \cap \hat{I}) = N(\emptyset)$. We can choose U open, $z_0 \in U$ so that $U \cap N(I) \subset N(\emptyset)$. To see this, assume not, i.e. assume there exists a sequence $\{z_n\}$, such that $z_n \in N(I)\backslash N(\emptyset)$ and $z_n \to z_0$, then $z_0 \in cl(N(I)\backslash N(\emptyset))$ hence $z_0 \in N(\emptyset)$. Thus $U \subset N(\hat{I})$. If $U \subset N(\hat{I})$ then $\sigma|_U = \tau_0|_U$. But $\bar{N}(\hat{I})$ is a neighborhood of $N(\hat{I})$ by construction, thus we can choose $U \subset N(I)$.

<u>Case 3.</u> ($z_0 \in N(\emptyset)$). Let U be open, $z_0 \in U$. By the previous examples we are only concerned with $z \in U \cap (N(I)\backslash N(\emptyset))$. But again U can be chosen such that $\sigma|_U = \tau_0|_U$.

We are now ready to perform the induction step. Let $I \subset \mathfrak{T}_k$ be an interval in P. Assume that $(N(I \cup \hat{I}), N(\hat{I}))$ is a regular index pair of I. We need to show that the same holds for any interval contained in $\mathfrak{T}_{k+1}$. As before $(\bar{N}(P), \bar{N}_\epsilon(\mathfrak{T}_{k+1}))$ is a regular index pair for $M(T_{k+1})$. Let $\Pi \in L_{k+1}$. Define

$$N(\Pi) = \{z \in N_\epsilon(\mathfrak{T}_{k+1}) \mid \ t \geq 0,\ z \cdot t \in N(\Pi)\} \backslash N(\mathfrak{T}_k).$$

Let $I \subset \mathfrak{T}_{k+1}$ be an interval in P. Then there exists a unique interval $J \subset \mathfrak{T}_k$ such that $I = \bigcup_{i=1}^{q} \Pi_i \cup J$ where $\Pi_i \in L_{k+1}$ for $i = 1,\ldots,q$. Define $N(I) = \bigcup_{i=1}^{q} N(\Pi_i) \cup N(J)$.

Let $\tau_k\colon \overline{N(P)} \to [0,\infty]$ be the function in definition 1.9 for the index pair $(\overline{N(P)}, N(\mathfrak{T}_k))$. Define $\sigma\colon N(I \cup \hat{I}) \to [0,\infty]$ by

$$\sigma(z) = \begin{cases} \sup\{t > 0 \mid z \cdot [0,t] \subset N(I \cup \hat{I}) \backslash N(\hat{I}) & \text{if } z \in N(I \cup \hat{I}) \backslash N(\hat{I}) \\ 0 & \text{if } z \in N(\hat{I}). \end{cases}$$

The proof that σ is continuous is as for the previous case. ∎

We finish this subsection by computing the Conley index for a hyperbolic critical point.

<u>Example 1.12.</u> Given $\dot{z} = f(z)$ a differential equation in R^n, let z_0 be a <u>hyperbolic</u> rest point of the resulting flow, (i.e. $f(z_0) = 0$ and all the eigenvalues of $Df(z_0)$ have non-zero real part). There exists a neighborhood, U, of z_0 with coordinates such that for any $\zeta \in U$ the flow can be

written as $\zeta \cdot t = (\zeta_1 e^t, \ldots, \zeta_k e^t, \zeta_{k+1} e^{-t}, \ldots, \zeta_n e^{-t})$ where $\zeta = 0$ corresponds to z_0. Define $N_1 = [-\epsilon_1, \epsilon_1] \times \ldots \times [-\epsilon_n, \epsilon_n] \subset U$ and $N_0 = \{\pm\epsilon_1\} \times \ldots \times \{\pm\epsilon_k\} \times [-\epsilon_{k+1}, \epsilon_{k+1}] \times \ldots \times [-\epsilon_n, \epsilon_n] \subset \partial N_1$ then (N_1, N_0) is an index pair for $S = \{z_0\}$. Thus $h(S) \sim \Sigma^k$ a pointed k-sphere and

$$H_j\,(h(S); \mathbf{Z}_2) \cong H_j(N_1, N_0; \mathbf{Z}_2) = \begin{cases} \mathbf{Z}_2 & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$$

1.2 Connection Matrix

Let S be an isolated invariant set. Then, in general, there are various Morse decompositions and orderings thereof, which one can associate with S, and for each possible Morse set there is a topological invariant, namely, the Conley index. The connection matrix is a method for organizing this information. *To be more precise, given* S *and a regular index filtration* $\mathcal{N} = \{N(I)\}$, the connection matrix extracts some of the information determined by $H_*(N(I \cup \hat{I}), N(\hat{I}))$, i.e. the homology of the index for each possible Morse set of S. The idea behind the connection matrix is surprisingly simple and elegant. Unfortunately, because there are so many possible index pairs, constructing the connection matrix is, in general, quite complicated. Therefore, before proceeding with the general definitions we give the following elementary but important example of a connection matrix.

Let $N_0 \subset N_1$ and $N_0' \subset N_1'$. A map between pairs $f:(N_1,N_0) \to (N_1',N_0')$ means that $f: N_1 \to N_1'$ continuously and $f(N_0) \subset N_0'$.

<u>Definition 1.13.</u> Let S be an isolated invariant set with a Morse decomposition $\{M(a),M(b)\}$. If $b > a$ then $M(a)$ is called an <u>attractor</u> and denoted, A, while $M(b)$ is called a <u>repeller</u> *and denoted,* A^*. *Together* (A,A^*) is called an <u>attractor-repeller pair</u> for S.

One can check that if $I \subset P$ is an attracting interval then $M(I)$ is an attractor and $M(P\backslash I)$ is the corresponding repeller. Furthermore, if (I,J) is an adjacent pair of intervals then $M(I) = A$, $M(J) = A^*$ is an attractor

repeller pair in $M(I \cup J)$.

Given an attractor-repeller pair (A,A^*) of S there exists a regular index filtration $N_0 \subset N_1 \subset N_2$ such that (N_2,N_0), (N_1,N_0), and (N_2,N_1) are regular index pairs for S, A, and A^* respectively. This leads to the sequence of maps

$$(N_1,N_0) \xrightarrow{i} (N_2,N_0) \xrightarrow{j} (N_2,N_1)$$

where i and j are inclusion maps. Passing to homology one has the long exact sequence

$$\cdots \to H_n(N_1,N_0) \xrightarrow{i_n} H_n(N_2,N_0) \xrightarrow{j_n} H_n(N_2,N_1) \xrightarrow{\Delta_n} H_{n-1}(N_1,N_0) \to \cdots$$

or equivalently

$$\begin{tikzcd}
H_*(h(A)) \arrow[rr, "i_*"] & & H_*(h(S)) \arrow[dl, "j_*"] \\
& H_*(h(A^*)) \arrow[ul, "\Delta"] &
\end{tikzcd} \tag{1.1}$$

Δ is the connection matrix for this example.

<u>Example 1.14.</u> Consider the case where (A,A^*) is an attractor-repeller pair for S and $C(A^*,A) = \emptyset$. Then there exists a regular index filtration $N_0 \subset N_1$, $N_0 \subset N_2$ such that $N_1 \cap N_2 = N_0$ and $(N_1 \cup N_2, N_0)$, (N_1,N_0), and (N_2,N_0) are regular index pairs for S, A, and A^*, respectively. Now

consider the maps

$$\begin{tikzcd}
(N_1,N_0) \arrow[r, "i"] \arrow[dr, "k"] & (N_1 \quad N_2,\ N_0) \arrow[d, "j"] \\
& (N_1 \cup N_2,\ N_0)
\end{tikzcd}$$

Passing to homology, one can check that k_*: $H(h(A)) \longrightarrow H(h(S))$ is an isomorphism, thus i_* is an injection and j_* a surjection. Exchanging N_1 and N_2 gives that the corresponding k_*: $H(h(A^*)) \longrightarrow H(h(S))$ is an isomorphism and hence, i_* is an injection and j_* a surjection. Thus in the long exact sequence (1.1) Δ is a zero map. Thus we have proved

<u>Theorem 1.15.</u> If (A,A^*) is an attractor-repeller pair in S and $S = A \cup A^*$ then the flow defined boundary map Δ: $H_*(h(A)) \to H_*(h(A))$ is trivial.

In application the contrapositive will be used.

<u>Corollary 1.16.</u> If Δ is not a trivial map then $S \neq A \cup A^*$, i.e. $C(A^*,A) \neq \emptyset$.

The connection matrix has been shown to exist if one uses homology (or cohomology) with field co-efficients. For our purposes it will suffice to use the field $\mathbb{Z}_2$. Also, since we are only interested in the homology of the index of $M(I)$ and not the homology of $M(I)$ we will write $H(I)$ or $H(M(I))$ in place of $H_*(h(M(I)); \mathbb{Z}_2)$.

Example 1.17. Let S be an isolated invariant set such that $H(S) = \bar{0}$, i.e. $H_i(S) = 0$ for all i. Assume that A and A^* are hyperbolic critical points with Conley index Σ^k and Σ^{k+1} respectively. From the long exact sequence, we have that Δ_{k+1}: $H_{k+1}(A^*) \to H_k(A)$ is an isomophism. Thus by Corollary 1.16 there exists a connecting orbit from A^* to A. Because Σ^k and Σ^{k+1} have only one non-zero homology group and because we are using $\mathbf{Z}_2$ coeficients we can write Δ: $H_*(A) \oplus H_*(A^*) \to H_*(A) \oplus H_*(A^*)$ in the form of a matrix as

$$\Delta = \begin{array}{cc} & \begin{array}{cc} A & A^* \end{array} \\ \begin{array}{c} A \\ A^* \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array} .$$

With this example in mind we now begin defining the connection matrix. Let I be an interval in P, then

$$\Delta(I): \bigoplus_{i \in I} H(i) \to \bigoplus_{i \in I} H(i)$$

Since $H(i) = H_*(h(M(i); \mathbf{Z}_2)$, it is a graded vector space and thus $\bigoplus_{i \in I} H_*(i)$ is a graded vector space. Thus $\Delta(I)$ is assumed to be a linear map and is represented by a matrix

$$\Delta(I) = \begin{bmatrix} \Delta(I)_{i,j} \end{bmatrix} : \begin{bmatrix} \vdots \\ H(j) \\ \vdots \end{bmatrix} \to \begin{bmatrix} \vdots \\ H(i) \\ \vdots \end{bmatrix}$$

where each $\Delta(I)_{ij}$ is a linear map from $H_*(j)$ to $H_*(i)$.

<u>Definition 1.18</u> (i) $\Delta(I)$ is <u>strictly upper triangular</u> if $\Delta(I)_{ij} = 0$ for $i \leq j$.

(ii) $\Delta(I)$ is a <u>boundary map</u> if $\Delta(I)_{ij}$ is of degree -1 and $\Delta(I)^2 = 0$.

Let I and J be intervals in P. Define $C\Delta(I) = \bigoplus_{i \in I} H(i)$ and $\Delta(J,I)$: $C\Delta(J) \to C\Delta(I)$ where

$$\Delta(J,I) = \left[\Delta(P)_{ij} \right]$$

such that $i \epsilon I$ *and* $j \epsilon J$. *If* $\Delta(P)$ *is a strictly upper triangular* boundary map then given I an interval of P one can define a chain complex $(C\Delta(I), \Delta(I))$. Of course, this chain complex generates homology groups which we denote $H\Delta(I)$. If (I,J) is an adjacent pair of intervals then one can define the exact sequence

$$0 \longrightarrow C\Delta(I) \xrightarrow{i} C\Delta(J \cup I) \xrightarrow{j} C\Delta(J) \longrightarrow 0$$

where i and j are the obvious inclusion and projection maps. Passing to homology we get the long exact sequence

$$\cdots \longrightarrow H_k\Delta(I) \longrightarrow H_k\Delta(I \cup J) \longrightarrow H_k\Delta(J) \xrightarrow{\overline{\Delta}(J,I)} H_{k-1}\Delta(I) \longrightarrow \cdots \qquad (1.2)$$

<u>Definition 1.19.</u> An upper triangular boundary map $\Delta(P)$ is a <u>connection matrix</u> for the Morse decomposition $\{M(\pi) \mid \pi \epsilon (P,>)\}$ of S if there exists isomorphisms $\Phi(I)\colon H\Delta(I) \to H(I)$ for any interval I in P such that:

(i) For every $\pi \epsilon P$, $\Phi(\pi)\colon H\Delta(\pi) \to H(\pi)$ is the identity

(ii) For every adjacent pair of intervals (I,J) the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & H\Delta(I) \arrow[r] \arrow[d, "\Phi(I)"] & H\Delta(I \cup J) \arrow[r] \arrow[d, "\Phi(I \cup J)"] & H\Delta(J) \arrow[r, "\bar{\Delta}(J,I)"] \arrow[d, "\Phi(J)"] & H\Delta(I) \arrow[r] \arrow[d, "\Phi(I)"] & \cdots \\
\cdots \arrow[r] & H(I) \arrow[r] & H(I \cup J) \arrow[r] & H(J) \arrow[r, "\Delta(J,I)"] & H(I) \arrow[r] & \cdots
\end{tikzcd}$$

where the top line is (1.2)

Franzosa [3] showed that using field co-efficients a connection matrix exists for any isolated invarient set S with Morse decomposition $\mathcal{M} = \{M(\pi) \mid \pi \epsilon (P,>)\}$. It should not be assumed that the connection matrix is unique, in fact, for many interesting examples it is not. Reineck [7] proved the following uniqueness theorem.

<u>Theorem 1.20</u>. Let $\{M(\pi) \mid \pi \epsilon P\}$ be a Morse decomposition of S such that each $M(\pi)$ in a hyperbolic critical point. If $W^u(M(\pi))$ $W^s(M(\pi))$ for all $\pi \neq \pi'$ then the connection matrix is unique with respect to the flow defined partial order.

The obvious question at this point is what kind of phenomena results in a non-unique connection matrix? There is still no satisfactory answer to this. However, the following example is enlightening.

INSERT FIGURE 2.

<u>Example 1.21.</u> Assume we have a parameterized family of flows whose phase plane portrait at values -1,0, and +1 are as shown in figure 2. Theorem 1.20 says that the connection matrices CM(1) and CM(-1) for the Morse decompositions M_1 and M_{-1}, respectively are unique. It is easy to check that they can be written as

$$CM(1) = \begin{array}{c c} & \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{array}{c} M_1 \\ M_2 \\ M_3 \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}, \quad CM(-1) = \begin{array}{c c} & \begin{array}{ccc} M_1 & M_2 & M_3 \end{array} \\ \begin{array}{c} M_1 \\ M_2 \\ M_3 \end{array} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.$$

What about CM(0)? Franzosa [3] showed that CM(-1) and CM(1) are both connection matrices for M_0 and that they are the only possible connection matrices. The non-uniqueness comes about because in M_0, 3 and 1 are not adjacent under the flow induced partial ordering and because we have a non-transverse saddle-saddle connection which can perturb to $M_{\pm 1}$.

We can ask another question, given CM(±1), is it possible to determine CM(0)? Again, in a general setting the answer is not clear (see Reineck [7]). However, for the classification project addressed in this paper it is sufficient

to be able to answer the following question. Given CM(1) and M_0 is it possible to determine CM(-1)? With suitable restrictions the answer is yes, as will be seen by the use of connection matrices of special systems called transition systems.

A final comment on the isomorphisms $\Phi(I)$: $H\Delta(I) \to H(I)$. Since $H\Delta(I)$ is generated by the chain complex $(C\Delta(I), \Delta(I))$, the isomorphism $\Phi(I)$ imposes algebraic restrictions on $\Delta(I)$. These algebraic restrictions will be referred to as the rank condition.

2.1. Assumptions

As the introduction indicated, we are really only concerned with finding solutions to the non-linear system of ordinary differential equations

$$\dot{x}_i = y_i \qquad (\mathcal{S}(\Theta))$$

$$\dot{y}_i = \Theta y_i - D_i V(x)$$

In order to apply the connection matrix to this problem we need a compact invariant set S, i.e. we want the set of bounded solutions to $(\mathcal{S}(\Theta))$, denoted by $S\mathcal{S}(\Theta)$ to be compact. The following two assumptions guarantee this.

(A1) There exists Q_0 such that if $Q < Q_0$ then the level surfaces $\{x \mid V(x) = Q\}$ are convex.

(A2) $\lim\limits_{\|x\|\to\infty} V(x) = -\infty$.

It should be mentioned that (A1) can be weakened without losing the fact that $S\mathcal{S}(\Theta)$ is compact. (See Conley [1]).

(A3) V has only non-degenerate critical points. These will be denoted by M_i, $i = 1,\dots,q$ and let $V(M_{i+1}) < V(M_i)$.

This assumption merely simplifies the presentation of results. Notice that M_1 is the absolute maximum of V.

An important assumption that might go un-noticed is in the P.D.E. originally considered. A more general form of (0.1) is

$$x_T = D\Delta x + V(x) \tag{2.1}$$

where D is a diagonal matrix with entries λ_i. Again, restricting our attention to traveling wave front solutions reduces (2.1) to the system of O.D.E.'s

$$\begin{aligned} x_i &= y_i \\ \dot{y}_i &= {}^1/_{\lambda_i}(\theta y_i - V_i(x) \end{aligned} \tag{2.2}$$

The important difference between ($\mathcal{S}(\theta)$) and (2.2) is that for ($\mathcal{S}(\theta)$) there exists a global Lyapanov function, something which in general does not occur for (2.2). The Lyapanov function is used to determine the Morse decomposition of $S\mathcal{S}(\theta)$ and to limit the number of possible connection matrices for various wave speeds θ. This is not to say that the techniques developed here are of no use in (2.1). Rather, one will have to pay closer attention to the structure of the Morse sets, and some of the questions asked at the end of Section 1.2 will have to be better understood, before the connection matrix leads to a classification scheme.

2.2. Basic Results for Traveling Wave Systems

This section translates simple analytic results for the traveling wave system $\mathcal{S}(\Theta)$ into the language needed to apply the Conley index and the connection matrix. Unless it is important to specify the wave speed we let $\mathcal{S} = \mathcal{S}(\Theta)$ and denote the set of bounded solutions to $\mathcal{S}$ by $\mathcal{SS}$. Though slightly different from the standard definition we call $H: \mathbb{R}^n \to \mathbb{R}$ a Liapanov function for a system of O.D.E,'s if either $\frac{dH}{dt} \geq 0$ or $\frac{dH}{dt} \leq 0$ (but not both) along all solutions of the O.D.E..

Proposition 2.1. If $\Theta \neq 0$ then $H(x,y) = \frac{1}{2} \langle y,y \rangle + V(x)$ is a Liapanov function for $\mathcal{S}$.

Proof. On solutions of $\mathcal{S}$ one has that

$$\frac{dH}{dt} = \langle y,\dot{y} \rangle + \langle \nabla V,y \rangle = \Theta \langle y,y \rangle \qquad \blacksquare$$

Corollary 2.2. If $\Theta > 0$ then $\frac{dH}{dt} \geq 0$ and if $\Theta < 0$ then $\frac{dH}{dt} \leq 0$ along solutions.

If $\Theta = 0$ then $\mathcal{S}(0)$ reduces to a Hamiltonian system with Hamiltonian function, H. It is the author's opinion that an understanding of the set of bounded solutions to the Hamiltonian system should give information on bounded solutions of $\mathcal{S}(\Theta)$ for Θ sufficiently small and, vice versa,

given any sequence $\theta_n \to 0$ for which $S\mathcal{S}(\theta_n)$ is understood one should be able to draw conclusions about the structure of the set of bounded solutions to the Hamiltonian problem. This question will be taken up in a future paper (see Mischaikow [6]).

It is easily checked that

Proposition 2.3. The only critical points of $\mathcal{S}$ are $\{(M_i, 0)\}_{i=1}^{q}$.

Proposition 2.5. For $\theta \neq 0$, $S\mathcal{S}(\theta)$ consists of the critical points $\{(M_i,0)\}_{i=1}^{q}$ and heteroclinic orbits connecting these critical points. Furthermore, given an $M_i \to M_j$ connection then:

(a) $\theta > 0$ implies $V(M_j) > V(M_i)$ i.e. $j < i$.

(b) $\theta < 0$ implies $V(M_j) < V(M_i)$ i.e. $j > i$.

Sketch of proof. This follows from proposition 2.1 and an easy computation showing that the only orbits along which H is constant are the critical points. (a) and (b) are restatements of corollary 2.2, i.e. if $\theta > 0$ then the "energy", H, is increasing and decreasing if $\theta < 0$. ∎

Definition 2.6. Let $\eta(M_i)$ be the number of negative eigenvalues of $D^2V(M_i)$.

Proposition 2.7. If $\eta(M_i) = k$ and $\theta > 0$ then the dimension of the unstable manifold at $(M_i,0)$ is $2n$-k. If $\theta < 0$ then the dimension of the unstable manifold is k.

<u>Proof.</u> Let $A = D^2V = \left[\frac{\partial^2 V}{\partial x_i \, \partial x_j}\right]$. Since A is a symmetric matrix, there exists B and B^{-1} orthogonal real matrices such that $B^{-1}AB = \bar{A}$ where $\bar{A}$ is a diagonal matrix with non-zero real entries $\lambda_1, \ldots, \lambda_n$. Notice that

$$\begin{bmatrix} 0 & I \\ \bar{A} & \theta I \end{bmatrix} = \begin{bmatrix} B^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ A & \theta I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}.$$

Thus the eigenvalues can be found by solving

$$\det \begin{bmatrix} xI & -I \\ -\bar{A} & (x-\theta)I \end{bmatrix} = \det \begin{bmatrix} xI & -I \\ 0 & \frac{x^2-\theta x-\lambda i}{x} \end{bmatrix}$$

$$= x^n \prod_{i=1}^{n} \frac{x^2-\theta x-\lambda_i}{x}$$

$$= \prod_{i=1}^{n} (x^2-\theta x-\lambda_i) = 0 .$$

This gives $x_i^{\pm} = \frac{1}{2}(\theta \pm \sqrt{\theta^2 - 4\lambda_i})$.

For $\theta > 0$:

If $\lambda_i > 0$ then $\sqrt{\theta^2+4\lambda_i} > \theta$, hence $x_i^+ > 0$ and $x_i^- < 0$.

If $\lambda_i < 0$ then $\sqrt{\theta^2+4\lambda_i} < \theta$, hence $x_i^{\pm} > 0$.

For $\theta < 0$:

If $\lambda_i > 0$ then $\sqrt{\theta^2+4\lambda_i} > |\theta|$, hence $x_i^+ > 0$ and $x_i^- < 0$.

If $\lambda_i < 0$ then $\sqrt{\theta^2+4\lambda_i} < |\theta|$, hence $x_i^{\pm} < 0$. ∎

Restating these results in the language of Conley gives:

<u>Theorem 2.8.</u> (a) $S\mathcal{S}(\Theta)$ is an isolated invariant set with a Morse decomposition $\{(M_i,0)\}_{i=1}^{q}$ for $\Theta \neq 0$. If $\Theta > 0$ then $1 < 2 < \cdots < q$ is always an admissable ordering. If $\Theta < 0$ then $q < q\text{-}1 < \cdots\cdot < 1$ is always an admissable ordering.

(b) Let $\eta(M_i) = k$. If $\Theta > 0$ then $h(M_i,0) \sim \Sigma^{2n-k}$. If $\Theta < 0$ then $h(M_i,0) \sim \Sigma^k$.

(c) $h(S\mathcal{S}(\Theta)) \sim \Sigma^n$.

<u>Proof.</u> (a) This follows directly from propositions 2.4 and 2.5.

(b) This follows from proposition 2.7 and example 1.12.

(c) Let $N = \{(x,y) \mid V(x) \geq Q_0$ and $||y|| \leq K\}$. In the proof of proposition 2.4, Conley shows that for K large enough, N is an isolating neighborhood of $S\mathcal{S}(\Theta)$. Now define a homotopy $W\colon \mathbb{R}^n \times I \to \mathbb{R}$ satisfying:

(i) $W(x,0) = V(x)$

(ii) If $x \in \mathbb{R}^n$ such that $V(x) \leq Q_0$ then $W(x,s) = V(x)$ for all $s \in [0,1]$

(iii) $W(x,1) = U(x)$ where U has a non-degenerate, unique critical point, P. Clearly, P is an absolute maximum since $\lim_{||x||\to\infty} U(x) = \lim_{||x||\to\infty} V(x) = -\infty$. Now define

$$\dot{x} = y$$

$$\dot{y} = \Theta y - \nabla U(x).$$

By proposition 2.5, the set of bounded solutions is P. But since $U=V$ for $\{x \mid V(x) = Q_0\}$, $h(P) \sim h(S\mathcal{S}(0))$. $\eta(P) = n$ hence (b) implies (c). ∎

It is important to notice that (a) does not imply that the total orderings given are the only possible orderings on the Morse decomposition. In fact, as will be shown, the flow induced partial order in much weaker.

Simple substitution gives:

Proposition 2.9. The transformation $\bar{\Theta} = -\Theta$, $\bar{Y} = -y$, $\tau = -t$ leaves $S(\Theta)$ invarient.

The importance of this proposition is that it allows us, for the time being, to restrict our attention to $S(\Theta)$ where $\Theta > 0$. In fact there exists a strong symmetry about $\Theta = 0$. If $z(t) \in SS(\Theta)$ with $\Theta > 0$ is a connection $M_i \rightarrow M_j$ then there exists a solution $\bar{z}(t) \in SS(-\Theta)$ which is a connection $M_j \rightarrow M_i$. This symmetry can be exploited to translate information from the Hamiltonian system to the traveling wave systems (see Mischaikow [6])

From now on, unless explicitly stated otherwise, it is assumed that $\Theta > 0$.

Proposition 2.10. Let M_i and M_j be different local maxima of V. If $\Theta >> 1$ is large enough then there does not exist a connection $M_i \rightarrow M_j$.

Proof. Let $z(t) = (x(t), y(t))$ be a solution to $S(\Theta)$. The total change in energy, H, along $z(t)$ is given by $\int_{-\infty}^{\infty} \frac{dH}{dt}(z(t))\, dt$. Let z be a curve on the unstable manifold of M_i. Since $\frac{dH}{dt} \geq 0$ it must be that for all $t \in \mathbb{R}$

$$H(z(t)) = H(x(t), y(t)) = \tfrac{1}{2}\langle y(t), y(t)\rangle + V(x(t)) \geq V(M_i).$$

M_i is a local maximum thus there is an $\epsilon > 0$ such that for all x where $\|M_i - x\| < \epsilon$ and $x \neq M_i$, $V(x) < V(M_i)$. In particular, there exists $\delta > 0$ and t_0 such that if $z(t)$ is a connecting orbit from $(M_i,0)$ to $(M_j,0)$ then $V(x(t_0)) = V(M_i) - \delta$, i.e. $\langle y(t_0), y(t_0)\rangle \geq 2\delta$. Since y is continuous there exists $K > 0$ such that $K < \int_{-\infty}^{\infty} \langle y,y\rangle dt$. Thus

$$V(M_j) - V(M_i) = \int_{-\infty}^{\infty} \frac{dH}{dt}\, dt = \theta \int_{-\infty}^{\infty} \langle y, y\rangle dt > \theta K$$

But $V(M_j) - V(M_i)$ is fixed while $\theta K \to \infty$ as $\theta \to \infty$. ∎

The final result of this section says that the structure of $S\mathcal{S}(\theta)$ becomes fixed for θ sufficiently large.

<u>Proposition 2.11.</u> For fixed V, there exists $\theta >> 1$ such that there exists a connecting orbit from $(M_i,0)$ to $(M_j,0)$ satisfying $\mathcal{S}(\theta)$ if and only if there exists a corresponding solution from $(M_i,0)$ to $(M_j,0)$ satisfying $\dot{x} = \nabla V(x)$.

<u>Sketch of Proof.</u> Let ∇_x denote gradient with respect to x. Then $\mathcal{S}(\theta)$ can be written as $\dot{x} = y$, $\dot{y} = \theta y - \nabla_x V(x)$. Let $x = {}^{\xi}/_{\theta}$ then $\mathcal{S}(\theta)$ becomes

$$\dot{\xi} = \theta y$$

$$\dot{y} = \theta(y - \nabla_{\xi} V({}^{\xi}/_{\theta})) .$$

For $\theta >> 1$ it was implied in the proof of the previous proposition that if $z(t) = (x(t), y(t))$ is a bounded solution to $\mathcal{S}$ then $\|y\|$ must remain

small, hence $\|\dot{y}\|$ must remain small, thus $\|y - \nabla x_i V(\xi/\Theta)\| \to 0$. Therefore we are looking for solutions to $\dot{x} = {}^1/\theta\ V_x(x)$. But since we want non-constant bounded solutions to S we need to reparameterize t. Let $\tau = {}^t/_\Theta$ and $' = {}^d/_{dt}$ then $x' = \nabla_x V(x)$. ∎

2.3. Transition and Perturbation Systems

Our classification scheme depends on being able to use θ as a parameter. With this in mind we introduce the transition system

$$\dot{x}_i = y_i$$

$$\dot{y}_i = \theta y_i - D_i V(x) \qquad\qquad T(\theta_0,\theta_1)$$

$$\dot{\theta} = \epsilon(\theta - \theta_0)(\theta - \theta_1) \qquad 1 >> \epsilon > 0, \quad \theta_1 < \theta_0$$

Let the set of bounded solutions to $T(\theta_0,\theta_1)$ be denoted by $ST(\theta_0,\theta_1)$. Again, when no confusion arises we shall drop the θ_0 and θ_1. Unless otherwise stated assume $\theta_0 > \theta_1 > 1$. Figure 3 shows what the phase portrait of $T(\theta_0,\theta_1)$ is like. Notice that at $\theta = \theta_1$ we have the system $S(\theta_1)$ and at $\theta = \theta_0$ we have $S(\theta_0)$.

INSERT FIGURE 3

Lemma 2.12. The only critical points of $T(\theta_0,\theta_1)$ are $\{(M_i,0,\theta_0)\}$ and $\{(M_i,0,\theta)\}$ $i=1,\ldots,q$.

To save writing, when θ_0 and θ_1 are known let $M_i = (M_i,0,\theta_0)$ and $\bar{M}_i = (M_i,0,\theta_1)$. Furthermore, $M_i \to \bar{M}_j$ denotes a solution, $Z(t)$, of $T(\theta_0,\theta_1)$ such that $w(z) = \bar{M}_j$ and $w^*(z) = M_1$. Let $H(x,y,\theta) = H(x,y)$.

Lemma 2.13. On solutions of T, $\left.\frac{dH}{dt}\right|_{t=t_0} = \Theta(t_0) <y(t_0), y(t_0)> \geq 0$, i.e. H is a Liapanov function for the transition system.

Lemma 2.14. ST is compact.

Proof. If $(z(t), \Theta(t)) \in ST$ then $\Theta_1 \leq \Theta(t) \leq \Theta_0$. The proof of proposition 2.4 shows that for every fixed Θ, there exists $K(\Theta) > 0$ such that $N(\Theta) = \{(x,y_1 | V(x) \geq Q_0$ and $\|y\| \leq K(\Theta)\}$ is an isolating neighborhood of $SS(\Theta)$. Let $K = \max_{\Theta\epsilon[\Theta_1-\epsilon,\Theta_0+\epsilon]} K(\Theta)$. Then one easily checks that for $N = \{(x,y) | V(x) \geq \Theta_0, \|y\| \leq K\}$, $N \times [\Theta_1-\epsilon,\Theta_0+\epsilon]$ is an isolating neighborhood for $ST(\Theta_0,\Theta_1)$. ∎

Lemma 2.15. If $\zeta(t) = (x(t),y(t),\Theta(t))$ is a non constant bounded solution of $T(\Theta_0,\Theta_1)$ *then* $\zeta(t)$ *is a heteroclinic orbit of one of the following forms.*

(a) $M_i \to M_j$, $i > j$ and $\Theta(t) = \Theta_0$ $\forall t \in \mathbb{R}$

(b) $\bar{M}_i \to \bar{M}_j$, $i > j$ and $\Theta(t) = \Theta_1$ $\forall t \in \mathbb{R}$

(c) $M_i \to \bar{M}_j$, $i \geq j$. Furthermore $i=j$ if and only if $\zeta(t) = (M_i,0,\Theta(t))$ $\forall t \in \mathbb{R}$

Proof. (a) and (b) are restatements of proposition 2.5.

(c) If $\Theta(t) \in (\Theta_1,\Theta_0)$ then $\dot{\Theta} < 0$ hence $\lim_{t\to\infty} \zeta(t) = \bar{M}_j$ and $\lim_{t\to-\infty} \zeta(t) = M_i$. Since $\frac{dH}{dt} \geq 0$, $V(M_i) \leq V(M_j)$ thus $i \geq j$. Clearly, if $\zeta(0) = (M_i,0,\Theta(0))$ then $\zeta(t) = (M_i,0,\Theta(t))$ for all t. On the other hand, if there exists t_0 where $\zeta(t_0) = (x(t_0), y(t_0), \Theta(t_0))$ and $(x(t_0), y(t_0)) \neq (M_k,0)$ for $k=1,\dots,q$ then the total change in H over $\zeta(t)$ is $\int_{-\infty}^{\infty}\Theta(t)<y(t), y(t)>dt > 0$. Thus $M_i \neq M_j$. ∎

We can collect the information of the previous lemmas as follows:

<u>Theorem 2.16.</u> (a) $S\mathcal{T}(\Theta_0,\Theta)$ is an isolated invariant set with a Morse decomposition $\{M_i\} \cup \{\bar{M}_{\bar{i}}\}$ and $i,\bar{i} = 1,\dots,q$. If $>_0$ and $>_1$ are admissable orderings for the Morse decompositions of $S\mathcal{S}(\Theta_0)$ and $S\mathcal{S}(\Theta_1)$, respectively, then an admissable ordering for $S\mathcal{T}(\Theta_0,\Theta_1)$, $>$, is given by

$$
\begin{array}{lll}
i > j & \text{if} & i >_0 j \\
\bar{i} > \bar{j} & \text{if} & \bar{i} >_1 \bar{j} \\
i > \bar{j} & \text{for all} & i\ ,\ \bar{j}\ .
\end{array}
$$

(b) If $\eta(M_i) = k$ then $h(M_i) \sim \Sigma^{2n-k+1}$ and $h(\bar{M}_i) \sim \Sigma^{2n-k}$.

(c) $h(S\mathcal{T}) \sim \bar{0}$.

<u>Proof.</u> (a) This follows from lemmas 2.14 and 2.15.

(b) The unstable manifold of $\bar{M}_i$ has not been changed while that of M_i has been increased in dimension by 1. Now apply example 1.12, proposition 2.7 and theorem 2.8.

(c) One can either compute this directly, i.e. N , the isolating neighborhood, has been given above, thus one can determine N_0 and check that $h(N/N_0,*) \sim \bar{0}$ or one can notice that $S\mathcal{T}(\Theta_0,\Theta_1)$ can be "continued" (see Conley [1] or Salamon [8]) to a flow without critical points and hence $h(S\mathcal{T}) \sim \bar{0}$. ∎

As will be seen the transition system is helpful if one knows $\mathcal{S}(\Theta)$ for some particular value of Θ. Given any potential function, V, however, it is

not clear how to find a Θ for which $S(\Theta)$ can be analyzed. To some extent we can get around this problem if there exists a V_0 for which $S_0(\Theta)$ is known and if there exists a reasonable homotopy from V_0 to V. With this in mind we make the following definition.

<u>Definition 2.16.</u> $V: \mathbb{R}^n \times [-\delta,1+\delta] \to \mathbb{R}$, $\delta > 0$ is a <u>critical point preserving smooth parameterized family</u> of potential functions if for all $s \epsilon [-\delta,1+\delta]$:

(i) $V(x,s) = V_s(x) \in C^2(\mathbb{R}^n \times [-\delta,1+\delta],\mathbb{R})$

(ii) V_s has q non-degenerate critical points denoted by $M_i(s)$ $i=1,\ldots,q$.

(iii) $V_s(M_i(s)) = V_0(M_i(0))$ and $\eta(M_i(s)) = \eta(M_i(0))$

(iv) $V_s(x) = V_0(x)$ if $x \in \{x \mid V_0(x) < Q_0\}$.

Let V be a critical point preserving family of potential functions. Then we have a <u>Perturbation system</u> given by

$$\dot{x}_i = y_i$$

$$\dot{y}_i = \Theta y_i - D_i V_s(x) \qquad\qquad P$$

$$\dot{s} = -\epsilon\, s(s-1) \qquad 0 < \epsilon << 1 \,,\ \Theta >> 1$$

Let $M_i = (M_i(0),0,0)$ and $\bar{M}_i = (M_i(1),0,1)$. Let SP denote the set of bounded solutions to P. As far as the index theory is concerned P and T are similar systems thus we have the following theorem.

<u>Theorem 2.17.</u> (a) SP is an isolated invarient set with a Morse decomposition $\{M_i\} \cup \{\bar{M}_{\bar{i}}\}$ for $i,\bar{i} = 1,\ldots,q$. If $>_0$ and $>_1$ are admissable

orderings for the Morse decomposition of $\mathcal{SS}$ for V_0 and V_1, respectively, then an admissable ordering for $S\mathcal{P}$, $>$, is given by

$$i > j \quad \text{if} \quad i >_0 j$$

$$\bar{i} > \bar{j} \quad \text{if} \quad \bar{i} >_1 \bar{j}$$

$$i > \bar{j} \quad \text{for all} \quad i, \bar{j} .$$

(b) If $\eta(M_i(0)) = k = \eta(M_i(1))$ then $h(M_i) \sim \Sigma^{2n-k+1}$ and $h(\bar{M}_i) \sim \Sigma^{2n-k}$.

(c) $h(S\mathcal{P}) \sim \bar{0}$.

However, as in the previous cases, we want slightly more detailed information as to the nature of the possible heteroclinic orbits.

<u>Proposition 2.18.</u> Let $\zeta(t) = (x(t), y(t), s(t))$ be a solution to $\mathcal{P}$. Then there exists a solution such that $\lim_{t\to\infty} \zeta(t) = \bar{M}_i$ and $\lim_{t\to -\infty} \zeta(t) = M_i$.

<u>Proof.</u> For fixed s, consider the system

$$\dot{x}_i = y_i$$

$$\dot{y}_i = \theta y_i - D_i V_s(x) .$$

$(M_i(s),0)$ is a hyperbolic critical point for this system. Let $\eta(M_i(s)) = k$. (Notice that k is independent of s). Via proposition 2.7 and example 1.12 one has that under a suitable change of coordinates there is an isolated

neighborhood $N(s)$ of $(M_i(s),0)$ with exit set $N_0(s)$ as in Figure 4. Now consider the system Θ with $\epsilon << 1$.

INSERT FIGURE 4.

Since V is a smooth family of potential functions we can define a set N which contains the arc $(M_i(s),0,s)$ for $s \in [{}^{-\delta}/_2,\ 1+{}^{\delta}/_2]$ and furthermore for fixed s, N restricts to $N(s)$ with $N_0(s)$ the exit set for N restricted to s. Let $N_0 = \bigcup_s N_0(s) \cup N(1+{}^{\delta}/_2)$. One can check that (N,N_0) is an index pair for some isolated invariant set, S. Furthermore, $h(S) = \bar{0}$. Now refering to example 1.17 we see that the connection matrix for S is

$$\begin{array}{c} \\ \bar{M}_i \\ M_i \end{array} \begin{array}{c} \begin{array}{cc} \bar{M}_i & M_i \end{array} \\ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array} .$$

In otherwords, there exists a connection from M_i to $\bar{M}_i$. ■

This proposition as it stands is not global enough for our purposes. We are interested in the set of connections from M_i to $\bar{M}_i$ which are contained in all of SP not just N. It might be possible that there is another connection from M_i to $\bar{M}_i$ which lies outside of N. It is to eliminate this possibility that the restriction $\Theta >> 1$ is included.

Proposition 2.19. Let $\zeta(t) = (x(t),\ y(t),\ s(t))$ be a solution of P such that $\lim_{t\to\infty} \zeta(t) = \bar{M}_i$ and $\lim_{t\to-\infty} \zeta(t) = M_i$. If $\Theta >> 1$ is sufficiently large then ζ lies in N as defined in proposition 2.18.

<u>Proof.</u> If we had a Liaponov function the theorem would be easy, e.g. lemma 2.15 (c). However, if $H(x,y,s) \equiv \frac{1}{2}\langle y,y\rangle + V(x,s)$ then along solutions of P

$$\frac{dH}{dt} = \Theta \langle y,y\rangle + \frac{\partial V_s}{\partial s}\dot{s}$$

is not a Liaponov function. Never-the-less, along $\zeta(t)$ we must have

$$0 = \int_{-\infty}^{\infty} \frac{dH}{dt}\,dt = \Theta \int_{-\infty}^{\infty} \langle y(t),\ y(t)\rangle dt + \int_{-\infty}^{\infty} \frac{\partial V_s}{\partial s}\dot{s}\ dt\ .$$

If $\zeta(t)$ leaves N for some values of t, say $t \in (a,b)$, then there exists a lower bound K_1 such that

$$0 < K_1 < \int_a^b \langle y,y\rangle\ dt < \int_{-\infty}^{\infty} \langle y,y\rangle\ dt$$

thus

$$0 < \Theta \int_{-\infty}^{\infty} \langle y,y\rangle\ dt = -\int_{-\infty}^{\infty} \frac{\partial V}{\partial s}\dot{s}\ dt\ .$$

But for $s \in [0,1]$ there exists a maximum for $\left|\frac{\partial V}{\partial s}s\right|$, call it K_2. Then

$$0 < \Theta K_1 < K_2 \int_{-\infty}^{\infty} \frac{ds}{dt}\,dt = K_2$$

Letting $\Theta \to \infty$ makes the inequality impossible. ∎

Corollary 2.20. If $\bar{M}_i$ and M_i are an adjacent pair in an admisable ordering of a Morse decomposition of SP or ST then the connection matrix for $(M_i, \bar{M}_i)$ is

$$\begin{array}{cc} & \begin{array}{cc} \bar{M}_i & M_i \end{array} \\ \begin{array}{c} \bar{M}_i \\ M_i \end{array} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{array} .$$

3.1 Connection Matrix for $S\mathcal{S}(\Theta)$.

We begin with a rough sketch of what are the possible connection matrices for $S\mathcal{S}(\Theta)$. It is always assumed that the set of Morse sets for the Morse decomposition of $S\mathcal{S}(\Theta)$ is $\{(M_i,0)\ i=1,\ldots,q$. Again let $M_i = (M_i,0)$. As was mentioned in Section 1.2 there need not be a unique connection matrix associated to this Morse decomposition. This motivates the following definition.

Definition 3.1. Θ is a standard wave speed for $\mathcal{S}$ if there exists a unique connection matrix, $CM(\Theta)$, for the isolated invarient set $S\mathcal{S}(\Theta)$.

Recall that if two Morse sets, M_i and M_j, are adjacent i.e. (i,j) is an adjacent pair, then the ij entry in the connection matrix is flow determined and hence uniquely determined. Thus for Θ to be a standard wave speed it is sufficient that, if $h(M_j) \sim \Sigma^{k+1}$ and $h(M_i) \sim \Sigma^k$ then M_i and M_j are adjacent Morse sets. This is only a slightly weaker condition than that of theorem 1.20 but suffices since $\Delta_{ij} = 0$ if $h(M_j) \sim \Sigma^k$, $h(M_i) \sim \Sigma^\ell$ and $k - \ell \neq 1$.

In order to describe the typical connection matrix, $CM(\Theta)$, it is useful to partition $\{M_i\}$ into subsets of the same index. The following notation will be used throughout the paper. Partition $\{1,\ldots,q\}$ into subsets J_k where $M_i \in J_k$ if and only if $h(M_i) \sim \Sigma^{2n-k}$. Let μ_k be the cardinality of J_k. Notice that it is possible for $J_k = \emptyset$ for some values of k. However, $J_n \neq \emptyset$ since V always has an absolute maximum and hence $M_1 \in J_n$.

<u>Example 3.2.</u> The condition that $V(M_{i+1}) < V(M_i)$ imposes mild restrictions on the set of possible partitions. For example, consider $V\colon \mathbb{R} \to \mathbb{R}$ with five critical points. In this case $n=1$, hence $k=0,1$ and $q=5$. There are only two possible partitions $J_1 = \{1,2,3\}$ and $J_0 = \{4,5\}$ or $J_1' = \{1,2,4\}$ and $J_2' = \{3,5\}$. The partitions can be realized by potential functions V and V', respectively. See figure 5.

INSERT FIGURE 5

Recall that the connection matrix is a degree -1 homomorphism. Thus $\Delta_{ji}\colon H(i) \to H(j)$ is zero unless $i \in J_k$ and $j \in J_{k+1}$. Hence for any $\Theta \neq 0$,

$$CM(\Theta) = \begin{array}{c} \\ J_n \\ J_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ J_1 \\ J_0 \end{array} \begin{array}{c} \begin{array}{cccccc} J_n & J_{n-1} & \cdots & & J_1 & J_0 \end{array} \\ \left[\begin{array}{cccccc} 0 & A_n(\Theta) & 0 & \cdots & & \\ & 0 & A_{n-1}(\Theta) & 0 & \cdots & \\ & & 0 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & 0 & A_1(\Theta) \\ & & & & & 0 \end{array}\right] \end{array} \tag{3.1}$$

where (i) $A_k(\Theta)$ is a $\mu_k \times \mu_{k-1}$ matrix

(ii) $A_k(\Theta) : \bigoplus_{i \in J_{k-1}} H(i) \to \bigoplus_{j \in J_k} H(j)$

(iii) $A_k(\Theta) \circ A_{k-1}(\Theta) = 0$.

If no confusion can arise let $A_i = A_i(\Theta)$.

<u>Proposition 3.3.</u> The rank of $CM(\Theta)$ is $\frac{1}{2}(q-1)$ and hence q, the number of critical points of V, is odd.

<u>Proof.</u> The rank condition implies that

$$H\Delta(\{1,\dots,q\}) \simeq H(h(S\mathcal{S}(\Theta))) \simeq H(\Sigma^n)$$

by theorem 2.8. But $H\Delta(\{1,\dots,q\})$ is generated by the chain complex $(\bigoplus_{i=1}^{q} H(i), CM(\Theta))$ i.e.

$$H\Delta(\{1,\dots,q\}) \simeq {}^{\operatorname{Ker} CM(\Theta)}/_{\operatorname{Im} CM(\Theta)}.$$

Since $H(\Sigma^n)$ has a unique non-trivial 1-dimensional vector space it must be that $\operatorname{Ker} CM(\Theta) - \operatorname{Rank} CM(\Theta) = 1$. Now $CM(\Theta)$ is a $q \times q$ matrix hence the rank is $\frac{1}{2}(q-1)$. ∎

<u>Proposition 3.4.</u> For fixed V there exists $\Theta(V)$ such that, if $\Theta > \Theta(V)$ then $CM(\Theta) = CM(\Theta(V))$.

This proposition follows from proposition 2.11. We will denote $CM(\Theta(V))$ by $CM(\infty) = CM(\infty,V)$.

3.2 Transition Matrices

Given a potential function V and a wave speed θ_0 we have said nothing yet as to how to compute $CM(\theta_0)$. However, assume that $CM(\theta_0)$ is known, can we determine the set of possible $CM(\theta)$ for $\theta \in (0,\infty)$? To answer this we turn to the transition system $\mathcal{T}(\theta_0,\theta_1)$.

Let $CMT(\theta_0,\theta_1)$ be a connection matrix for $S\mathcal{T}(\theta_0,\theta_1)$. Theorem 2.16 implies that $CMT(\theta_0,\theta_1)$ is a $2q \times 2q$ matrix. Because the isolated invariant sets $S\mathcal{S}(\theta_0)$ and $S\mathcal{S}(\theta_1)$ are isolated invariant sets in $S\mathcal{T}(\theta_0,\theta_1)$ we can write

$$CMT(\theta_0,\theta_1) = \begin{bmatrix} CM(\theta_1) & T(\theta_0,\theta_1) \\ 0 & CM(\theta_0) \end{bmatrix}$$

where $CM(\theta_0)$ and $CM(\theta_1)$ are the connection matrices for $S\mathcal{S}(\theta_0)$ and $S\mathcal{S}(\theta_1)$, respectively. $T(\theta_0,\theta_1)$ is called a transition matrix from $CM(\theta_0)$ to $CM(\theta_1)$.

Proposition 3.5. Rank $CMT(\theta_0,\theta_1) = q$.

Proof. $H(h(S\mathcal{T}(\theta_0,\theta_1))) \simeq \bar{0}$ is trivial, hence Ker $CMT(\theta_0,\theta_1)$ = Rank $CMT(\theta_0,\theta_1)$. ∎

From now on we assume that θ_0 and θ_1 are standard wave speeds when we consider $\mathcal{T}(\theta_0,\theta_1)$. If it is clear from the context what θ_0 and θ_1, we let $T(\theta_0,\theta_1) = T$, $CMT(\theta_0,\theta_1) = CMT$, etc..

The Morse decomposition of $ST(\Theta_0,\Theta_1)$ is indexed by $\{1,\ldots,q,\bar{1},\bar{2},\ldots,\bar{q}\}$. There exists the obvious partition of this set given by $\{J_k\}_{k=0}^n$ and $\{\bar{J}_k\}_{k=0}^n$ where $i \in J_k$ if and only if $h(M_i) \sim \Sigma^{2n-k+1}$ and $\bar{j} \in \bar{J}_k$ if and only if $h(M_{\bar{j}}) \sim \Sigma^{2n-k}$. (See theorem 2.16).

<u>Proposition 3.6.</u> (a)

$$T = \begin{array}{cc} & \begin{array}{ccc} J_n & \cdots\cdots & J_0 \end{array} \\ \begin{array}{c} \bar{J}_n \\ \vdots \\ \vdots \\ \bar{J}_0 \end{array} & \begin{bmatrix} T_n & 0 & \cdots & \\ 0 & T_{n-1} & & \vdots \\ & & \ddots & 0 \\ \cdots & & 0 & T_0 \end{bmatrix} \end{array}$$

where T_k is a $\mu_k \times \mu_k$ matrix.

(b) $CM(\Theta_1) \circ T + T \circ CM(\Theta_0) = 0$

(c) T_k is upper triangular with diagonal entries equal to 1.

<u>Proof.</u> (a) Any element of T represents a degree -1 map from $H(j)$ to $H(\bar{i})$. Thus if t_{ij} is an element of T and $h(M_j) \sim \Sigma^{k+1}$ then $t_{ij} = 0$ unless $h(\bar{M}_i) \sim \Sigma^k$. By theorem 2.16 it must be that $\eta(M_i) = \eta(M_j)$ and thus $t_{ij} \in T_\ell$, $0 \leq \ell \leq n$.

(b) This is just a re-statement of the fact that $CMT(\Theta_0,\Theta_1)$ is a connection matrix and hence a boundary map, i.e. $CMT^2 = 0$.

(c) That T_k is upper triangular follows from lemma 2.15(c). In addition, since we are assuming Θ_0 and Θ_1 are standard wave speeds, M_i and $\bar{M}_i$ are adjacent Morse sets in the flow induced partial ordering. Hence, the fact that there exists a unique M_i to $\bar{M}_i$ connection implies that the diagonal entries are 1. ∎

The point of view that we want to adopt is that $CM(\theta_0)$ and T are known and $CM(\theta_1)$ is to be found.

Proposition 3.7. $CM(\theta_1) = T \circ CM(\theta_0) \circ T^{-1}$.

Proof. T is triangular with non zero diagonal entries and hence invertible. The result now follows from proposition 3.6(b). ∎

There are two problems with using arbitrary transition matrices. First, even a small number of critical points leads to a large number of transition matrices. Second, and more importantly, it is not always easy to gain information on the types of connections which exist from a general transition matrix (see Section 3.3). With this in mind we introduce the following class of special transition matrices.

Definition 3.7. An elementary transition matrix, $E(i,j)$, is a transition matrix of the form $I + \Delta_{ji}$ where I is the identity matrix and Δ_{ji} has only one non zero entry, the $ji^{\underline{th}}$.

Definition 3.8. A transition graph is a connected graph whose vertices are connection matrices for S and whose edges are transition matrices. Furthermore, if $CM(\theta_0)$ and $CM(\theta_1)$ are vertices connected by the edge $T(\theta_0,\theta_1)$ then

$$\begin{bmatrix} CM(\theta_1) & T(\theta_0,\theta_1) \\ 0 & CM(\theta_0) \end{bmatrix}$$

is a connection matrix for $ST(\Theta_0,\Theta_1)$. If each $T(\Theta_0,\Theta_1)$ is an elementary transition matrix we have an elementary transition graph.

Remark 3.9. This definition is purposely ambiguous. As will be seen, paths in the elementary transition graph classify the traveling wave solutions to (0.1) and as such there is a natural way to present the graph. For other applications, different presentations may prove to be more useful. In this paper, the vertices, i.e. the connection matrices, will be taken as unique though each vertex may have several names. To be more precise, assume Θ_0 is a standard wave speed, then $CM(\Theta_0)$ is unique and because of the continuity property of the connection matrix (See Franzosa [3]) there exists $\epsilon > 0$ such that for all $\Theta \in (\Theta_0 - \epsilon, \Theta_0 + \epsilon)$, $CM(\Theta) = CM(\Theta_0)$. Thus in the transition graph there will be the $CM(\Theta)$. However, this vertex could also be called $CM(\Theta')$ for any $\Theta' \in (\Theta_0 - \epsilon, \Theta_0 + \epsilon)$. If in addition there is another standard wave speed, Θ'', such that $|\Theta'' - \Theta|$ is large but $CM(\Theta'') = CM(\Theta)$ then the vertex corresponding to $CM(\Theta)$ could also be called $CM(\Theta'')$.

With regard to the edges, different edges may have the same name. For example,

$$\text{if} \quad \begin{bmatrix} B & T \\ 0 & A \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \bar{B} & T \\ 0 & \bar{A} \end{bmatrix}$$

are connection matrices for ST, but $B \neq \bar{B}$ and $A \neq \bar{A}$ then between vertices A and B, and $\bar{A}$ and $\bar{B}$ lie different edges, but both are called T.

For the purposes of classification we will always want to use a "maximal" elementary transition matrix. However, there are at least two ways to define maximal. The first relies completely on the algebra. Notice that combining (3.1), propostions 3.3, 3.5 and 3.6 gives a set of algebraic conditions which the connection matrices and transition matrices must satisfy. Furthermore, these conditions are completely determined by the partition $\{J_k\}_{k=1}^n$.

Definition 3.10. Given $\{J_k\}_{k=1}^n$, an algebraically maximal transition graph is a transition graph such that if $CM(\theta_0)$ is a vertex of the graph and

$$\begin{bmatrix} CM(\theta_1) & T \\ 0 & CM(\theta_0) \end{bmatrix}$$

is an algebraically permissable connection matrix for the abstract system ST then $CM(\theta_1)$ is a vertex of the graph and T is an edge connecting $CM(\theta_0)$ to $CM(\theta_1)$.

On the other hand, it might be that one has additional information which precludes cetain connection matices, CM, or certain transition matrices, T.

Definition 3.11. Given $\{J_k\}_{k=1}^n$ and apriori restrictions on the set of possible connection matrices, CM, and transition matrices, T, a realizable maximal transition graph is a transition graph such that if $CM(\theta_0)$ is a vertex of the graph and

$$\begin{bmatrix} CM(\Theta_1) & T \\ 0 & CM(\Theta_0) \end{bmatrix}$$

is a realizable connection matrix, $CMT(\Theta_0,\Theta_1)$, then $CM(\Theta_1)$ is a vertex and T is an edge connecting $CM(\Theta_0)$ and $CM(\Theta_1)$.

Assuming that we know $CM(\Theta_0)$ for some value $\Theta = \Theta_0$, we can construct the maximal transition graph by using propostion 3.11 and concatenating. In otherwords, given $CM(\Theta_0)$ we can generate a set of possible connection matrices, $CM = \{CM(\Theta_i) \mid i=1,\ldots,p\}$. An obvious question is, do we generate the same set of connection matrices if we restrict to elementary transition matrices? The answer is yes as will be shown in what follows.

Given elementary transition matrices, E_k, let

$$\prod_{k=1}^{n} E_k = E_n \circ E_{n-1} \circ \cdots \circ E_2 \circ E_1 \quad \text{and} \quad \prod_{k=n}^{1} E_k = E_1 \circ E_2 \circ \cdots \circ E_n.$$

Let

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i=j \\ 0 & \text{otherwise} \end{cases} .$$

It is simple to check that

<u>Lemma 3.12.</u> $E(i,j) = E(i,j)^{-1}$.

This says that in the case of elementary transition matrices, proposition 3.7 can be written as

$$CM(\Theta_1) = E(i,j)\, CM(\Theta_0)\ E(i,j)\ .$$

Thus, the following lemma will be useful.

Lemma 3.13. Let $A = [a_{k\ell}]$ then

$$E(i,j) \circ A = A + \sum_{\ell} (\delta_{ik}\ a_{k\ell})\ \Delta_{j\ell} \qquad \text{and}$$

$$A \circ E(i,j) = A + \sum_{k} (a_{k\ell}\ \delta_{ij})\ \Delta_{ki}\ .$$

Proposition 3.14. Let

$$T = \begin{bmatrix} T_n & & 0 \\ & \ddots\ T_k \ddots & \\ 0 & & T_1 \end{bmatrix}$$

be a transition matrix. Let $t_{ij} \in T_k$. Then $E(i,j) \circ T = T'$ where

$$T' = \begin{bmatrix} T_n & & 0 \\ & \ddots\ T_k' \ddots & \\ 0 & & T_1 \end{bmatrix}.$$

Proof. By the previous lemma $E(i,j) \circ T = T + \sum_{\ell} t_{i\ell}\Delta_{j\ell}$. But $t_{i\ell} = 0$ unless $t_{i\ell} \in T_k$. ∎

<u>Definition 3.15.</u> Let T be a connection matrix. An <u>elementary decomposition</u> of T is an ordered sequence $\{E_k = E(i_k,j_k)\}$ of elementary transition matrices where $T = \prod_k E_k$.

<u>Proposition 3.16.</u> Let T_k be a block in T, a transition matrix. Then either $T_k = I$ or there exists an elementary decomposition of T_k.

<u>Proof.</u> By the previous proposition we need only consider $E(i,j)$ such that $t_{ij} \in T_k$. Since T_k is a $\mu_k \times \mu_k$ matrix we will consider $E(i,j)$ to be a $\mu_k \times \mu_k$ matrix. (This merely simplifies the notation.) The proof of the proposition is by induction on the size of T_k.

Assume T_k is a 2×2 matrix. Then $T_k = I$ or $T_k = E(2,1)$. So assume T_k is an $n \times n$ matrix. Let $\tilde{T}_k = [\tilde{t}_{ij}]$ where $\tilde{t}_{ij} = t_{ij}$ for all $j < n$ and let $\tilde{t}_{in} = \delta_{in}$, $i=1,\dots,n$. Notice that $\tilde{T}_k$ satisfies the restrictions of proposition 3.6. Assume that there exists an elementary decomposition of $\tilde{T}$. Let

$$\hat{E}(n,i) = \begin{cases} E(n,i) & \text{if } t_{in} = 1 \\ I & \text{if } t_{in} = 0 \end{cases}$$

<u>Lemma 3.17.</u> $T_k = \left(\prod_{k=n-1}^{1} \hat{E}(n,k)\right) \circ \tilde{T}$.

<u>Proof.</u> The proof is by induction. If $\hat{E}(n,n-1) = I$ then $t_{n-1,n} = \tilde{t}_{n-1,n}$. If $\hat{E}(n,n-1) = E(n,n-1)$ then

$$\hat{E}(n,n-1) \circ \tilde{T} = \tilde{T} + \sum_{\ell} (\delta_{nk} \bar{t}_{k\ell})\, \Delta_{n-1,\ell}$$

$$= \tilde{T} + \sum_{\ell} (\bar{t}_{n\ell})\, \Delta_{n-1,\ell}$$

$$= \tilde{T} + \Delta_{n-1,n} .$$

Now after $r-1$ steps assume one has $\tilde{T} + \sum_{k=1}^{r-1} t_{n-k,n}\, \Delta_{n-k,n}$. Consider

$$E(n,n-r) \circ (\tilde{T} + \sum_{k=1}^{r-1} t_{n-k,n}\, \Delta_{n-k,n}) = E(n,n-r) \circ \tilde{T} + \sum t_{n-k,n}\, E(n,n-r)\, \Delta_{n-k,n}$$

$$= \tilde{T} + \sum_{\ell} (\delta_{nk} \bar{t}_{k\ell}) \Delta_{n-r,\ell} + \sum_{k=1}^{r-1} t_{n-k,n}\, \Delta_{n-k,n} + \sum_{k=1}^{r-1} t_{n-k,n}\, \Delta_{n-r,n}\, \Delta_{n-k,n}$$

$$= \tilde{T} + \sum_{\ell} \bar{t}_{n\ell}\, \Delta_{n-r,\ell} + \sum_{k=1}^{r-1} t_{n-k,n}\, \Delta_{n-k,n}$$

$$= \tilde{T} + \sum_{k=1}^{r} t_{n-k,n}\, \Delta_{n-k,n} \qquad \blacksquare$$

The lemma gives an elementary decomposition of T_k. ■

<u>Proposition 3.18.</u> Let T be a transition matrix. Then T has an elementary decomposition.

<u>Proof.</u> Let

$$T = \begin{bmatrix} T_n & & 0 \\ & \ddots \; T_k \; \ddots & \\ 0 & & T_1 \end{bmatrix} .$$

By propostion 3.14 if $E_m(k)$ is an elementary transition matrix in an elementary decomposition of T_k then $E_m(k)$ has no effect on T_j , $j \neq k$. Thus $T = \prod_{k=1}^{n} \prod_{m} E_m(k)$ where $\{E_m(k)\}$ is an elementary decomposition of T_k.

∎

It is important to notice that transition matrices do not have unique elementary decompositions.

Example 3.19. (a) Let $T = I$ then $T = \prod_k E_k E_k^{-1}$. Also $E = E \circ E \circ E$.

(b) Let $T = \begin{bmatrix} T_2 & 0 \\ 0 & T_1 \end{bmatrix}$ with elementary decompositions for T_2 and T_1 given by $\{E_k(2)\}$ and $\{E_k(1)\}$, respectively. Then

$$\prod_k E_k(2) \circ \prod_k E_k(1) = \prod_k E_k(1) \circ \prod_k E_k(2) .$$

(c) Let $T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ then $T = E(2,1) \circ E(3,2)$

$$= E(3,2) \circ E(2,1) \circ E(3,1) .$$

The last example suggests that elementary transition matrices do not commute. The next proposition makes this precise.

Proposition 3.20. $E(\ell,k) \circ E(i,j) = E(i,j) \circ E(\ell,k)$ if and only if $i \neq k$ and $j \neq \ell$.

Proof. $E(\ell,k) \circ E(i,j) = E(i,j) \circ E(\ell,k)$

$$E(\ell,k) = E(i,j) \circ E(\ell,k) \circ E(i,j)$$

$$I + \Delta_{k\ell} = (I + \Delta_{ji}) \circ (I + \Delta_{k\ell}) \circ (I + \Delta_{ji})$$

$$= I + \Delta_{k\ell} + \Delta_{k\ell}\Delta_{ji} + \Delta_{ji}\Delta_{k\ell} + \Delta_{ji}\Delta_{k\ell}\Delta_{ji} .$$

Thus we need to know the conditions under which

$$\Delta_{k\ell}\Delta_{ji} + \Delta_{ji}\Delta_{k\ell} + \Delta_{ji}\Delta_{k\ell}\Delta_{ji} = 0 .$$

This is the same as

$$\delta_{j\ell}\Delta_{ki} + \delta_{ik}\Delta_{j\ell} + \delta_{j\ell}\delta_{ik}\Delta_{ji} = 0$$

which holds if and only if $i \neq k$ and $j \neq \ell$. ■

Definition 3.21. $E(i,j) \boxplus E(\ell,k) = I + \Delta_{ji} + \Delta_{k\ell}$

Corollary 3.22. (a) $E(i,j) \boxplus E(\ell,i) \neq E(i,j) \circ E(\ell,i)$

(b) $E(\ell,i) \boxplus E(i,j) = E(\ell,i) \boxplus E(i,j)$.

Proof. (a) follows from the proposition. (b) By proposition 3.6, $\ell > i > j$ thus $\ell \neq j$. The rest follows from simple calculations. ■

Corollary 3.23. Recall that $\{J_k\}_{k=0}^{n}$ is a partition of $\{1,\dots,q\}$. Let $\ell \in J_k$ and $m \in J_{k'}$ where $k \neq k'$. Then

$$E(\ell,m) \boxplus E(i,j) = E(\ell,m) \circ E(i,j) = E(i,j) \circ E(\ell,m).$$

We now consider the types of paths that lie in the elementary transition graphs.

Proposition 3.24. Let

$$\begin{bmatrix} CM(\Theta_m) & T \\ 0 & CM(\Theta_0) \end{bmatrix}$$

be a connection matrix, $CMT(\Theta_0,\Theta_m)$. Let $T = \prod_{k=1}^{m} E_k$. Then

$$CM(\Theta_0) \xrightarrow{E_1} CM(\Theta_1) \xrightarrow{E_2} CM(\Theta_2) \longrightarrow \cdots \xrightarrow{E_m} CM(\Theta_m)$$

is a path in the elementary transition graph of $CM(\Theta_0)$.

Proof. By definition $CM(\Theta_k) = E_k CM(\Theta_{k-1}) E_k$. So we need only check that

$$CM(\Theta_m) = (\prod_{k=1}^{m} E_k)\; CM(\Theta_o)\; (\prod_{k=m}^{1} E_k) = TCM(\Theta_0)\; T^{-1} .$$ ∎

This proposition says that given any $CM(\Theta_0)$ and any transition matrix $TM(\Theta_0,\Theta_1)$, the connection matrix $CM(\Theta_1)$ lies in the elementary transition graph. Furthermore, $CM(\Theta_1)$ can be found by tracing the path in the graph determined by any elementary decomposition of $TM(\Theta_0,\Theta_1)$.

Corollary 3.25. If

$$\begin{bmatrix} CM(\Theta_1) & E \\ 0 & CM(\Theta_0) \end{bmatrix}$$

is a connection matrix where E is an elementary transition matrix then so is

$$\begin{bmatrix} CM(\Theta_0) & E \\ 0 & CM(\Theta_1) \end{bmatrix} .$$

Proof. Let $T = I$, recall $I = E \circ E$ and apply the proposition. ∎

This implies that the elementary transition graphs are not directed. In otherwords, a path along the graph can consist of

$$CM(\Theta_0) \overset{E}{\text{——}} CM(\Theta_1) \overset{E}{\text{——}} CM(\Theta_0) .$$

Proposition 3.26. Let $CM(\Theta) = A = [a_{ij}]$ and $E = E(i,j)$ be an elementary transition matrix. $A \overset{E}{\supset}$ is a subgraph of an elementary transition graph if and only if $0 = a_{ik} = a_{kj}$ for $k=1,...,q$.

Proof. Let $T = I$, recall $I = E \circ E$ and apply the propostion. ∎

3.3. Interpreting Transition Matrices

Consider the system $T(\theta_0,\theta_1)$, but let $\epsilon = 0$. Let $\underset{\theta}{\cdot}$ denote the flow generated by $S(\theta)$, then the flow on $\mathbb{R}^{2n} \times \mathbb{R}$ generated by $T(\theta_0,\theta_1)$ and denoted by $\cdot$ satisfies

$$(z,\theta) \cdot t = (z \underset{\theta}{\cdot} t,\theta) .$$

In otherwords, $\mathbb{R}^{2n} \times \{\theta\}$ is an invarient subset under the flow and the flow on this subset is determined by $S(\theta)$. Assume that for some value $\theta^* \epsilon (\theta_1,\theta_0)$ there exists a connecting orbit from $(M_i,0,\theta^*)$ to $(M_j,0,\theta^*)$ where $i,j \epsilon J_k$. Now let $0 < \epsilon < < 1$. Figure 6 suggest what the new flow looks like. In particular, one expects that there exists a connection $M_i \rightarrow \bar{M}_j$. One might in turn hope that $t_{ji} = 1$ where $t_{ji} \epsilon T(\theta_0,\theta_1)$.

INSERT FIGURE 6.

Since we are assuming $T(\theta_0,\theta_1)$ known, it would be nice if the following conjecture were true. Assume that for all $\epsilon \in (0,\epsilon_0)$,

$$CMT(\theta_0,\theta_1) = \begin{bmatrix} CM(\theta_1) & T(\theta_0,\theta_1) \\ 0 & CM(\theta_0) \end{bmatrix}$$

is a connection matrix for $ST(\theta_0,\theta_1)$. Let $t_{ji} = 1$, $i \neq j$ and $t_{ji} \epsilon T(\theta_0,\theta_1)$. Then there exists $\theta^* \epsilon (\theta_0,\theta_1)$ such that there is a connection $M_i \rightarrow M_j$, a solution to $S(\theta^*)$. Unfortunately, in this generality the conjecture is false.

To see why, recall example 3.19 (c) and consider figure 7 which gives a schematic representation of how the two different elementary decompositions could be realized by the flow. One now sees that if the connection $M_3 \to M_2$ preceeds the connection $M_2 \to M_1$ then in the flow with $\epsilon > 0$ it is possible for the connection $M_3 \to \bar{M}_1$ to occur.

INSERT FIGURE 7.

In order to state a correct version of the conjecture we make the following definition and restrictions. For fixed V. Let

$$W = W(V) = \{\theta > 0 \mid \theta \text{ is not a standard wave speed}\}.$$

From now on we only consider potential functions which satisfy the following two assumptions:

(A4) W is a discrete subset of $(0,\infty)$

(A5) Let $\theta^* \in W$, then there exists a unique i,j and k such that $i,j \in J_k$ and there is a unique connection $M_i \to M_j$ which is a solution to $\mathcal{S}(\theta^*)$.

<u>Definition 3.28.</u> Let $\theta_A > \theta_B > 0$. $\{\theta_A,\theta_B\}$ are an <u>adjacent pair of wave speeds</u> if θ_A and θ_B are standard wave speeds and $W \cap (\theta_B,\theta_A) = \theta^*$.

<u>Theorem 3.29.</u> (Reineck [7]) Suppose that for $\epsilon \in (0,\epsilon_0)$, M_1 and $\bar{M}_j$ are adjacent in the $\mathcal{T}(\theta_0,\theta_1)$ flow defined partial order and that $1 = t_{ji} \in T(\theta_0,\theta_1)$. Then there are $\theta_2,\ldots,\theta_k \in (\theta_1,\theta_0)$ and $M_i = M_{m_2},\ldots,M_{m_{k+1}} = M_j$ such

that $m_p >_p m_{p+1}$, $p=1,\dots,k$, where $>_p$ is the flow defined partial order in $S\mathcal{S}(\theta_p)$.

Proposition 3.30. Let $\{\theta_A,\theta_B\}$ be an adjacent pair of wave speeds. If

$$CMT(\theta_A,\theta_B) = \begin{bmatrix} CM(\theta_B) & T(\theta_A,\theta_B) \\ 0 & CM(\theta_A) \end{bmatrix}$$

is a connection matrix for $S\mathcal{T}(\theta_A,\theta_B)$ then $T(\theta_A,\theta_B) = E(i,j)$ an elementary transition matrix. The converse is also true. Given $\{\theta_A,\theta_B\}$ an adjacent pair with

$$\begin{bmatrix} CM(\theta_B) & E(k,\ell) \\ 0 & CM(\theta_A) \end{bmatrix}$$

the connection matrix for $S\mathcal{T}(\theta_A,\theta_B)$, then there exists $\theta^* \in (\theta_B,\theta_A)$ such that $\theta^* \in W$ and there exists a connection $M_k \to M_\ell$ a solution of $\mathcal{S}(\theta^*)$.

Proof. This is just a special case of theorem 3.29. ∎

Because we will work with elementary transition matrices rather than the more general transition matrices, we have not attempted to correct the conjecture in the most general sense possible. However, it may be worth noticing that for a transaction matrix T, the correspondence between $t_{ij} = 1$ and the existence of θ^* corresponding to the connection $M_j \to \bar{M}_i$ will occur only if

$$T = E_n \boxplus E_{n-1} \boxplus \cdots \boxplus E_1$$

where $\{E_k\}$ is the elementary decomposition of T realized by the flow.

Several comments are in order concerning (A4) and (A5). (A4) seems to be a generic assumption, though the author claims no proof of this. It would be interesting to have examples of potential functions, V, which fail (A4) in the following three ways.

1) W is dense on some interval in $(0,\infty)$.

2) W contains an interval.

3) W has a positive limit point. (There are many interesting cases where 0 is the limit point of W, namely those which have an infinite number of traveling wave solutions.)

(A5) also appears to be a generic assumption, however it raises different questions. Let $\{\theta_A,\theta_B\}$ be an adjacent pair and let θ^* be the unique element of $W \cap (\theta_B,\theta_A)$, but assume that there exists more than one heteroclinic solution to $\mathcal{S}(\theta^*)$ between critical points in the same partition. Furthermore let

$$T(\theta_A,\theta_B) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Do there exist perturbations of V which realized the diagrams in figure 7?

3.4 Transition Matrices for SP

It should not come as a surprise that the connection matrices for some potential functions are easier to determine than for other potential functions. The perturbation system, P, can be used to gain information about difficult to analyze potential functions from simple potential functions, i.e. ones with strong symmetries. Theorem 2.17 allows one to conclude that the connection matrix for SP is of the form

$$\begin{bmatrix} CM(1) & T \\ 0 & CM(0) \end{bmatrix},$$

where $CM(1)$ and $CM(0)$ are the connection matrices of the potential functions V_1 and V_0, respectively, at the wave speed $\theta >> 1$. T is again called the transition matrix.

Proposition 3.30. (a) $T = \begin{bmatrix} 0 & 0 & \cdots & \\ 0 & T_{n-1} & 0 & \\ & & \ddots & \\ & 0 & & 0 \\ & & 0 & T_0 \end{bmatrix}$ where T_k is a $\mu_k \times \mu_k$ matrix.

(b) T_k is upper triangular with diagonal entries equal to 1.

Proof. (a) For T_k, $k=0,\ldots,n-1$ the argument is the same as for proposition 3.6(a). In the case of T_n, however, proposition 2.10 says that for θ

sufficiently large there does not exist a connection $M_i \to M_j$ if $i,j \in J_n$. Now, if an element of T_n is non zero for all $\epsilon \in (0,\epsilon_0)$ then there exists at least one connecting orbit of the above type (see theorem 3.29) for some potential function V_s, $s \in (0,1)$.

(b) If there exist $t_{ij} \in T_k$ with $i > j$ then by theorem 3.29, for some value of s, $s \in (0,1)$, proposition 2.5(a) is contradicted for V_s. ∎

Notice that no assumption is made that the transition matrix be an elementary transition matrix. However if $t_{ij} \in T_n$ then there exists an elementary decomposition of T which does not contain $E(j,i)$.

4. Examples

This section consists of simple examples to demonstrate how the elementary transition graphs can be used to obtain information about the set of possible bounded solutions to $\mathcal{S}(\theta)$ for various values of θ. For the following discussion it must be kept in mind that if θ_A and θ_B are an adjacent pair of wave speeds then it is possible that $CM(\theta_A) = CM(\theta_B)$. (See remark 3.9)

Definition 4.1. For fixed V the path from θ_A to θ_B in the elementary transition graph is the path

$$CM(\theta_A) = CM(\theta_0) \xrightarrow{E_0} CM(\theta_1) \xrightarrow{E_1} \cdots \xrightarrow{E_k} CM(\theta_{k+1}) = CM(\theta_B)$$

where $\{\theta_i, \theta_{i+1}\}$ are adjacent pairs, $i=0,1,\ldots,k$.

Assumptions (A4) and (A5) assure that given $\theta_A > \theta_B > 0$ there exists a path from θ_A to θ_B. By proposition 3.4, for each V there exists a $CM(\infty,V)$ which can be taken as the starting point of the path. The total path of V is the maximal path beginning with $CM(\infty,V)$.

4.1 A Simple Potential Function

Let $V: \mathbb{R}^2 \to \mathbb{R}$ with $q=5$ and assume $J_0 = \emptyset$. Thus we are partitioning the set $\{1,2,3,4,5\}$ into J_2 and J_1. It can be shown that the only possible portions are $J_2 = \{1,2,3\}$ and $J_1 = \{4,5\}$ or $J_2' = \{1,2,4\}$ and $J_1' = \{3,5\}$. As the reader can check, in what follows the results for the partition J_2', J_1' are contained in the results for J_2, J_1. Hence we shall only consider the latter case. (3.1) says that a connection matrix for $S\mathcal{S}(\Theta)$ must be of the form

$$\mathrm{CM}(\Theta) = \begin{array}{c} \\ J_2 \\ J_1 \end{array} \!\!\begin{array}{c} \begin{array}{cc} J_2 & J_1 \end{array} \\ \begin{bmatrix} 0 & A_2(\Theta) \\ 0 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

where $*$ denotes an unknown entry. Proposition 3.3 implies that the rank of $\mathrm{CM}(\Theta)$ is 2 hence the rank of $A_2(\Theta)$ is 2. Since we are using $\mathbb{Z}_2$ coefficients there are 42 matrices $A_2(\Theta)$ which satisfy this restriction. As will be shown we can do much better. To save space we shall no longer write out the complete 5×5 matrix but rather the matrix $A_2(\Theta)$ and call it $\mathrm{CM}(\Theta)$.

As was remarked before, the perturbation system, P, can be used to determine the elementary transition graph for a general potential function if there exists a simple potential function which is related to it by a smooth critical point preserving homotopy. We call a potential function $\bar{V}: \mathbb{R}^n \to \mathbb{R}$ co-linear if all the critical points of V lie on a straight line. Since we are

considering $V\colon \mathbb{R}^2 \to \mathbb{R}$ and $J_0 = \emptyset$ it is easy to find a critical point preserving homotopy of V to $\bar{V}$ where $\bar{V}$ is co-linear. (The homotopy consists of sliding the critical points along the contour lines). Furthermore, this co-linear potential function $\bar{V}$ is equivalent, for our purposes, to a potential function mapping $\mathbb{R}$ to $\mathbb{R}$ For the moment then, assume $\bar{V}\colon \mathbb{R} \to \mathbb{R}$, $q=5$, $J_1 = \{1,2,3\}$ and $J_0 = \{4,5\}$. (Notice that the subscripts changed because the dimension of the system $S(\Theta)$ dropped from 4 to 2.). Proposition 3.4 gives the existence of $CM(\infty,\bar{V})$. Let

$$CM_L(\infty) = \{CM(\infty,\bar{V}) \mid \bar{V}\colon \mathbb{R} \to \mathbb{R},\ q=5,\ J_1 = \{1,2,3\} \text{ and } J_0 = \{4,5\}\}.$$

<u>Proposition 4.2.</u>

$$CM_L(\infty) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

<u>Proof.</u> Since $\bar{V}\colon \mathbb{R} \to \mathbb{R}$ the critical points of $\bar{V}$ must line up as follows

$$M_{i_1} \qquad M_{j_1} \qquad M_{i_2} \qquad M_{j_2} \qquad M_{i_3}$$

where $i_k \in J_1$ and $j_k \in J_0$. Using proposition 2.11 or Terman [9] one concludes that for Θ sufficiently large the only connections are the unique

connections $M_{j_1} \to M_{i_1}$, $M_{j_1} \to M_{i_2}$, $M_{j_2} \to M_{i_2}$ and $M_{j_2} \to M_{i_3}$. Considering all possible permentations of the M_i's and M_j's gives the result. ■

Proposition 3.6 implies that the set of possible elementary transition matrices is {(E(2,1), E(3,1), E(3,2), E(5,4)}. Since $\bar{V}: \mathbb{R} \to \mathbb{R}$ we can exclude E(5,4). Finally each M_{i_k} has a 1-dimensional stable manifold. In the case of M_{i_1} or M_{i_3} at least one of the orbits on the stable manifold is unbounded in backwards time. Thus the number of non zero entries in CM(Θ) must be less than or equal to 4. Using these restrictions we can generate the realizable maximal transition graphs shown in Figure 8. Because the Conley index and connection matrix is stable under perturbation (See Conley [1], Franzosa [3]) we have the following:

Theorem 4.3. Let $V: \mathbb{R}^n \to \mathbb{R}$ be sufficiently close, in a suitable metric, to the potential function $\bar{V}: \mathbb{R} \to \mathbb{R}$ discussed above. Then the total path of $\bar{V}$ lies in either $\bar{G}_1$, $\bar{G}_2$ or $\bar{G}_3$.

INSERT FIGURE 8.

An E(i,j) division of a transition graph G, are the subgraphs of G obtained by deleting all E(i,j) edges. In the case of the elementary transition graphs $\bar{G}_i$ i=1,2,3 an E(2,1) division always results in two disjoint subgraphs. We can denote the subgraphs by $\bar{H}_i$ and $\bar{L}_i$ where none of the vertices of $\bar{L}_i$ are elements of $CM(\infty)$.

For $\bar{V}: \mathbb{R} \to \mathbb{R}$ a simple shooting argument shows that there exists a unique wave speed Θ^* such that the connection $M_2 \to M_1$ is a solution to

$S(\Theta^*)$. Therefore, if $\Theta < \Theta^*$, $CM(\Theta)$ is a vertex of $\bar{L}_i$. If $CM(\infty,V)$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

then for $\Theta < \Theta^*$, $CM(\Theta)$ is a vertex of $\bar{L}_2$. $\bar{L}_2$ is a particularly simple graph. Notice that the two vertices differ only in the 2,4 and 2,5 entries. One can show (See Terman [9]) that given $\Theta_0 > 0$ it is not possible for $M_4 \to M_2$ or $M_5 \to M_2$ to be a solution to $S(\Theta)$ for all $\Theta \in (0,\Theta_0]$. This forces the path of V to keep alternating between the two vertices of $\bar{L}_2$ for $\Theta < \Theta^*$. Hence we can conclude that there exists an infinite number of elements of $W(\bar{V})$ for which an $M_3 \to M_2$ connection is a solution. In 4.2 we shall attempt to repeat this proof in order to show that there exists an infinite number of $M_3 \to M_2$ connections for a particular $V: \mathbb{R}^2 \to \mathbb{R}$ To see an alternative proof of the result for $\bar{V}$ see Terman [9].

We now return to the original problem, $V: \mathbb{R}^2 \to \mathbb{R}$ and show how the elementary transition graphs of V are related to those of $\bar{V}$. Because $\bar{V}: \mathbb{R} \to \mathbb{R}$, we were able to restrict the set of possible edges and vertices, and hence, the graphs $\bar{G}_i$ are not algebraically maximal. Dropping these restrictions, the graphs $\bar{G}_i$ i=1,2,3 become the algebraically maximal graphs G_j, j=1,2, shown in figure 9.

It still needs to be shown that G_j, j=1,2, are the only possible elementary transition graphs. But given V, it is related to some $\bar{V}: \mathbb{R}^2 \to \mathbb{R}$ a co-linear potential function via a critical point preserving family of potential functions. Thus $CM(\infty,V)$ and $CM(\infty,\bar{V})$ are the same or are connected by an E(5,4)

edge. Refering to G_j we get that $CM(\infty) = CM_L(\infty)$ where

$$CM(\infty) \equiv \{CM(\infty,v)\,|\,V\colon \mathbb{R}^2 \to \mathbb{R},\ q=5,\ J_2 = \{1,2,3\},\ \text{and}\ J_1 = \{4,5\}\}.$$

Thus we have

<u>Theorem 4.4.</u> Given $V\colon \mathbb{R}^2 \to \mathbb{R}$, $q=5$, $J_2 = \{1,2,3\}$ and $J_1 = \{4,5\}$, the total path of V lies in G_1, i=1,2.

It is worth noting that this theorem does have content to it. The algebraic restrictions allowed for the possibility of 42 connection matrices. Theorem 4.4 says that at most 24 are realizable for any system $S(\Theta)$.

INSERT FIGURE 9.

4.2. Existence of an Infinite Number of Traveling Waves

We are interested in showing how the results of 4.1 can be applied to a specific system. The goal is to show that there exists an infinite number of wave speeds for which an $M_3 \to M_2$ connection occurs. The proof is similar to that given for the 1-dimensional potential function, $\bar{V}$, of Section 4.1. The assumptions on V are made in order to emphasize the connection matrix techniques and to minimize the otherwise necessary computations. Let $V: \mathbb{R}^2 \to \mathbb{R}$ have contour lines as shown in figure 10.

As before we assume that V satisfies assumptions (A1) - (A5). In addition we assume the following.

(A6) K is a gradient line of V, i.e. if $x \in K$ then $\nabla V(x)$ is tangent to K.

(A7) The stable manifold of $(M_4,0)$ (note that it is one dimensional) projects to the right of L for all $\theta > 0$.

(A8) There exists a unique homoclinic orbit at $(M_2,0)$, for the system $\mathcal{S}(0)$.

(A9) Let $V(M_3) < h < V(M_2)$. Let $z(t) = (x(t), y(t))$ be a bounded solution to $\mathcal{S}(0)$ such that $H(z(t)) = h$. Then there exists $\{t_k\}_{k=1}^{\infty}$, $t_k \to \infty$ and $x(t_k) \cap K \neq \emptyset$, i.e. $x(t)$ crosses K infinitely often.

(A10) Fix θ. Then the unstable manifold of $(M_j,0)$ $j=4,5$ and the stable manifold of $(M_2,0)$ can intersect each other non-transversally along at most an odd number of connecting orbits.

INSERT FIGURE 10.

Theorem 4.5. Let V be a potential function as above, then there exists an infinite number of wave speeds for which an $M_3 \to M_2$ connection is a solution.

The rest of this section details the proof of this theroem. The above assumptions will be explained as they are used. We begin by using (A6) to prove the following proposition.

Proposition 4.6. $CM(\infty,V) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

Proof. One can construct a critical point preserving homotopy from V to $\bar{V}$, a co-linear potential function where the critical points are arranged as

$$\underset{|}{M_1} \text{———} \underset{|}{M_5} \text{———} \underset{|}{M_3} \text{———} \underset{|}{M_4} \text{———} \underset{|}{M_2} \quad .$$

Clearly,

$$CM(\infty,\bar{V}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} .$$

Notice that the homotopy can be performed without violating (A6) for any of the potential functions along the homotopy. But (A6) makes it impossible for an $M_5 \to M_4$ connection to occur for any wave speed. An E(5.4) is the only edge which can be realized in the perturbation system for this problem.

Thus, $CM(\infty, V) = CM(\infty, \bar{V})$. ∎

Corollary 4.7. The total path of V lies in the $E(5,4)$ division of G_2 which contains $CM(\infty, V)$.

Proof. As was mentioned above, (A6) implies that $E(5,4)$ cannot be an edge in the path of V. ∎

Proposition 4.8. There exists an odd number of wave speeds for which there exist $M_2 \to M_1$ connections.

For a proof of this the reader is refered to Mischaikow [6]. The result follows from (A8) in a non trivial manner, which requires comparing $CMT(-\Theta, \Theta)$ for small and large values of Θ. *For the* co-linear problem a simple shooting argument is sufficient to give this result. Clearly, this is not the case for the 4-dimensional system.

Corollary 4.9. There exists a wave speed Θ^* such that the path of V corresponding to $\Theta < \Theta^*$ lies in the graph:

$$\begin{tikzcd}
\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \arrow[r, dash, "E(3,2)"] \arrow[d, dash, "E(3,1)"] & \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \arrow[d, dash, "E(3,1)"] \\
\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix} \arrow[r, dash, "E(3,2)"] & \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}
\end{tikzcd}$$

Proof. By proposition 4.8 there exists a least wave speed θ^* such that an $M_2 \to M_1$ connection occurs. Because an odd number of E(2,1) edges lie on the total path of V, the portion of the path corresponding to $\theta < \theta^*$ must lie in L_2. (L_2 is defined in the equivalent manner as $\bar{L}_2$). Including the E(5,4) division of corollary 4.7 gives the desired graph. ∎

Conjecture. For $\theta < \theta^*$, CM(θ,V) is a vertex of $\bar{L}_2$.

A nice fact about this approach is that we are still able to prove theorem 4.5 without having to resolve this conjecture. By examining the graph in Corollary 4.9 one sees that a $M_3 \to M_2$ connection occurs if and only if either an $M_5 \to M_2$ connection occurs and an $M_4 \to M_2$ connection stops or an $M_5 \to M_2$ connection stops and an $M_4 \to M_2$ connection occurs. Thus we need to be able to show that $M_5 \to M_2$ and $M_4 \to M_2$ connections cannot persist for all $\theta \in (0,\theta^*)$.

With this in mind we define

$$Z_j: (\theta_A,\theta_B) \times [0,1] \to \mathbb{R}^4 \qquad j=4,5$$

a continuous map with the following properties.

$$Z_j(\theta) : \{\theta\} \times [0,1] \to \mathbb{R}^4$$

where $Z_j(\theta)\ (0) = (M_j,0)$, $Z_j(\theta)\ (1) = (M_2,0)$ and $Z_j(\theta)\ (0,1)$ corresponds to a connection $M_j \to M_2$ for the system $S(\theta)$.

If, for example, Θ is a standard wave speed and

$$CM(\Theta) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

then $Z_j(\Theta)$ can be defined. The interesting question is, given Θ, what is the maximal interval (Θ_A, Θ_B) on which Z_j can be defined.

Let $\Pi : \mathbb{R}^4 \to \mathbb{R}^2$ be the projection $\Pi(x,y) = x$. Let $\Theta \in (\Theta_A, \Theta_B)$. Define $\nu(Z_j(\Theta))$ to be the number of points in which $\Pi(Z_j(\Theta)[0,1])$ intersects K.

<u>Proposition 4.10.</u> $\nu(Z_j) \equiv \nu(Z_j(\Theta))$ for $\Theta \in (\Theta_A, \Theta_B)$ is well defined, i.e. ν is independent of Θ.

<u>Proof.</u> If $\Theta, \Theta' \in (\Theta_A, \Theta_B)$ and $\nu(Z_j(\Theta)) \neq \nu(Z_j(\Theta'))$ then there exists Θ'' such that $\Pi(Z_j(\Theta'')(0,1))$ is tangent to K. But $Z_j(\Theta'')(0,1)$ represents a solution to $\mathcal{S}(\Theta'')$. Since K is a gradient line, (A6), any solution whose projection under Π is tangent to K lies entirely on K. Thus $Z_j(\Theta'')(0) \neq (M_j,0)$. Contradiction. ∎

<u>Corollary 4.11.</u> $\nu(Z_5)$ is odd and $\nu(Z_4)$ is even.

<u>Proposition 4.12.</u> $\nu(Z_j(\Theta)) \to \infty$ as $\Theta \to \infty$.

<u>Proof.</u> Because $Z_j(\Theta)$ is a path from $(M_j,0)$ to $(M_2,0)$ there exists $\alpha^* \in (0,1)$ such that $H(Z_j(\Theta)(\alpha)) \in (V(M_3), V(M_2))$ for all $\alpha \in (\alpha^*,1)$. Thus for Θ sufficiently small $Z_j(\Theta)(\alpha^*+\epsilon, 1-\epsilon)$ $1 >> \epsilon > 0$ can be approximated by a

solution of $S(0)$. (A9) implies that $\Pi(Z_j(\Theta)(\alpha^*+\epsilon, 1-\epsilon)$ can be made to intersect K arbitrarily often for Θ sufficiently small. ∎

Corollary 4.13. If $Z_j(\Theta) : (\Theta_A,\Theta_B) \times [0,1] \to \mathbb{R}^4$ is as defined above then $\Theta_A > 0$.

This proves the theorem since (A10) implies that at each Θ_A the 2,j entry of the connection matrix must change.

References

[1] C.C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS, Regional Conf. Series in Math. 38 (1978).

[2] C.C. Conley and E. Zehnder, Morse Type Index Theory for Flows and Periodic Solns. for Hamiltonian Equations, *Comm. on Pure and Appl. Math.* Vol XXXVII, 207-253 (1984).

[3] R. Franzosa, *Index Filtrations and Connection Matrices for Partially Ordered Morse Decompositions*, Ph.D. Dissertation, Univer. of Wis. - Madison (1984).

[4] R. Franzosa, Index Filtrations and the Homolgy Index Braid for Partially Ordered Morse Decompositions, MRC Technical Summary Report #2810 (1985).

[5] J. Munkres, *Elements of Algebraic Topology*, Adison Wesley, 1984.

[6] K. Mischaikow, Standing Waves vs. Traveling Waves, in preparation.

[7] J. Reineck, *The Connection Matrix and the Classification of Flows Arising From Ecological Models*, Ph.D. Dissertation, Univ. of Wis.-Madison (1985).

[8] D. Salamon, Connected Simple Systems and the Conley Index of Isolated Invariant Sets, MRC Tech. Summary Report #2753 (1984).

[9] D. Terman, Directed Graphs and Traveling Waves, *Trans. AMS.* Vol 289, Number 2, pp. 809-847.

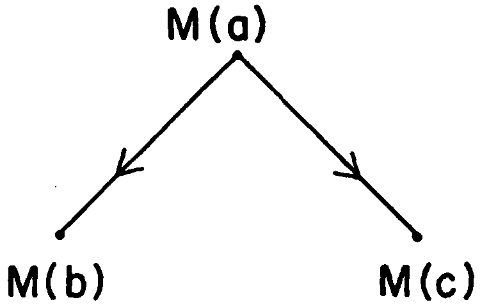


FIG. 1

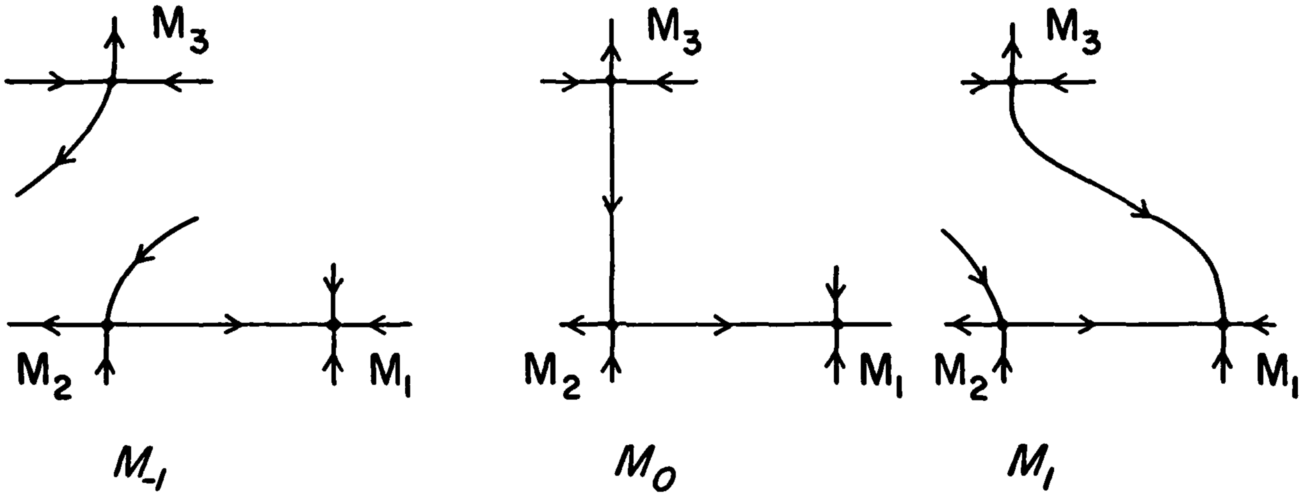


FIG. 2

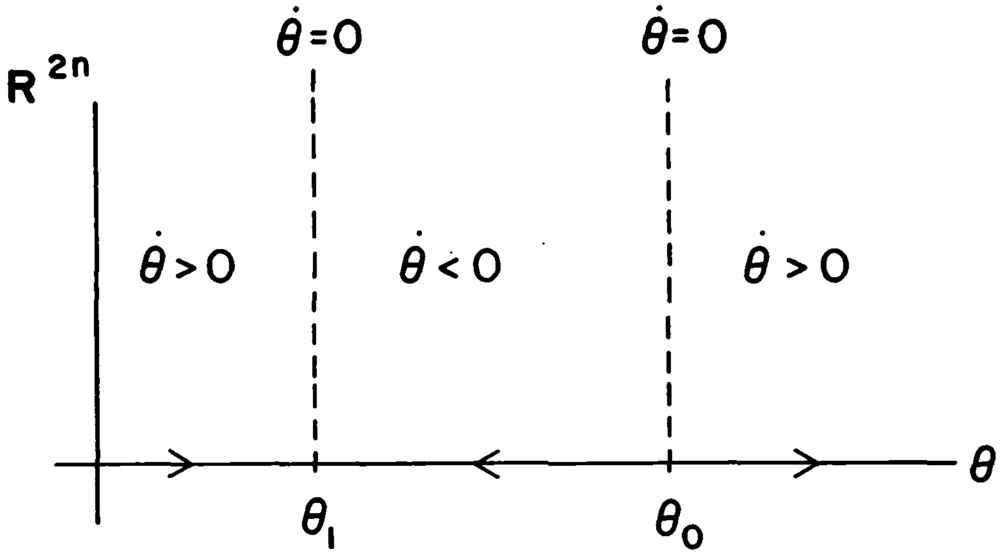

R^{2n}
$\dot\theta=0$
$\dot\theta=0$
$\dot\theta>0$
$\dot\theta<0$
$\dot\theta>0$
θ
θ_1
θ_0

FIG. 3

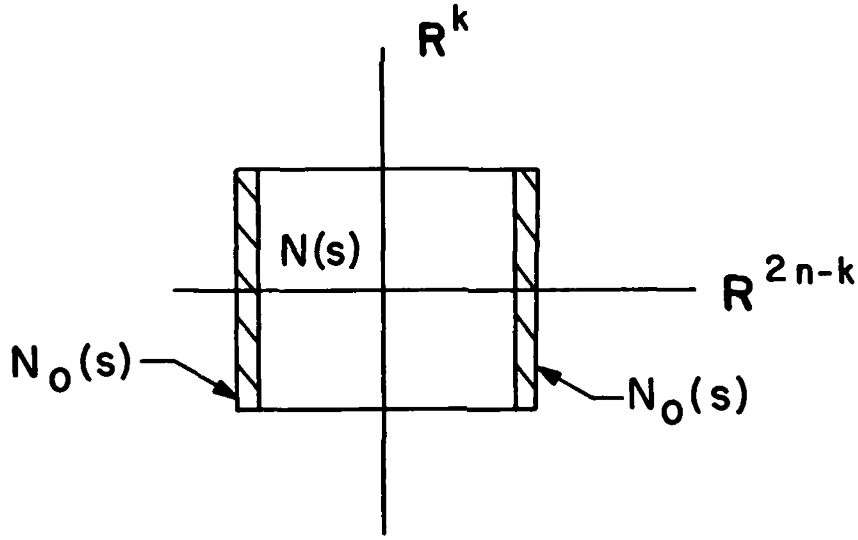

R^k
$N(s)$
R^{2n-k}
$N_0(s)$
$N_0(s)$

FIG. 4

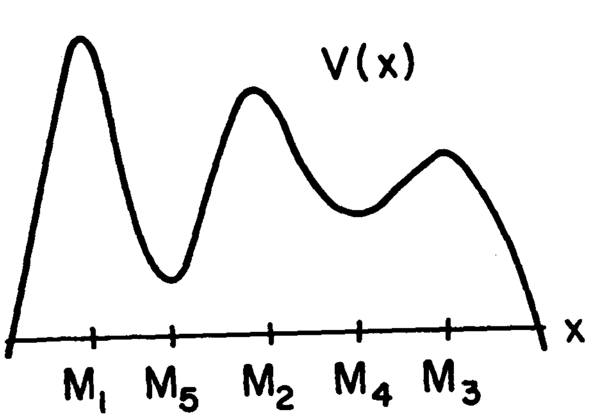

V(x)
x
M_1 M_5 M_2 M_4 M_3
V'(x)
x
M_1 M_3 M_2 M_5 M_4

FIG. 5

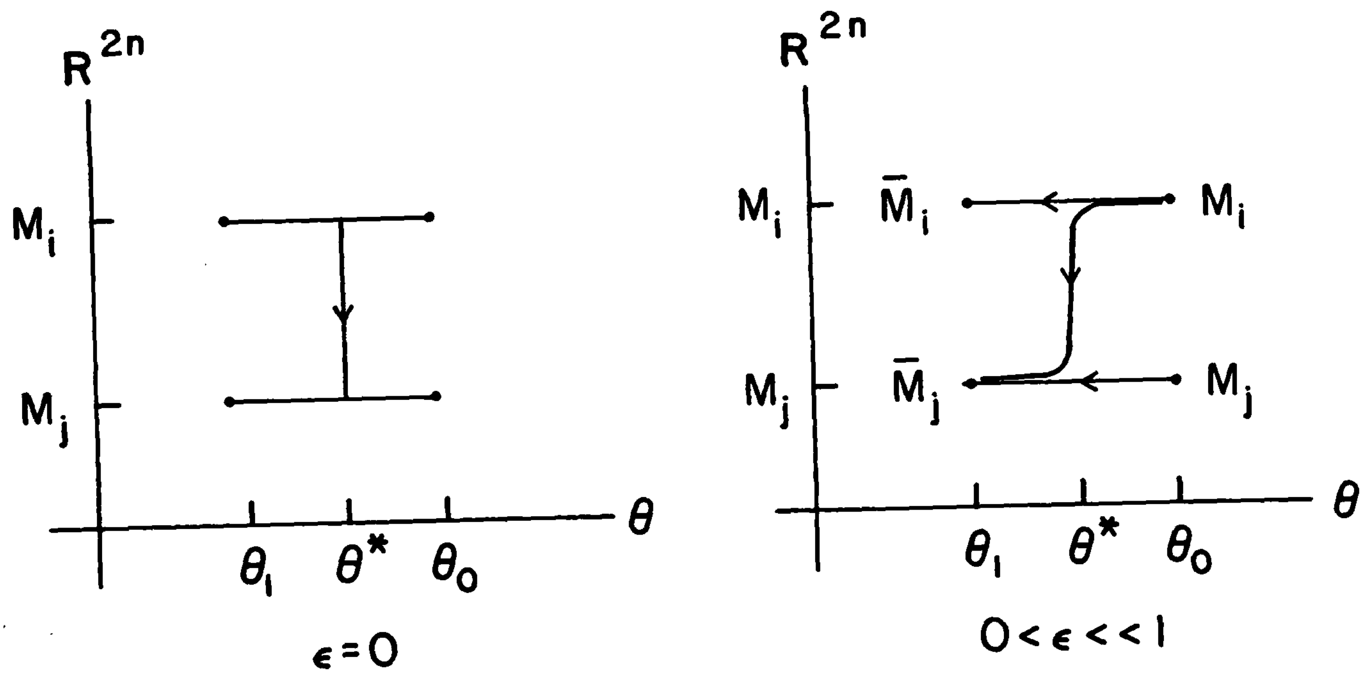
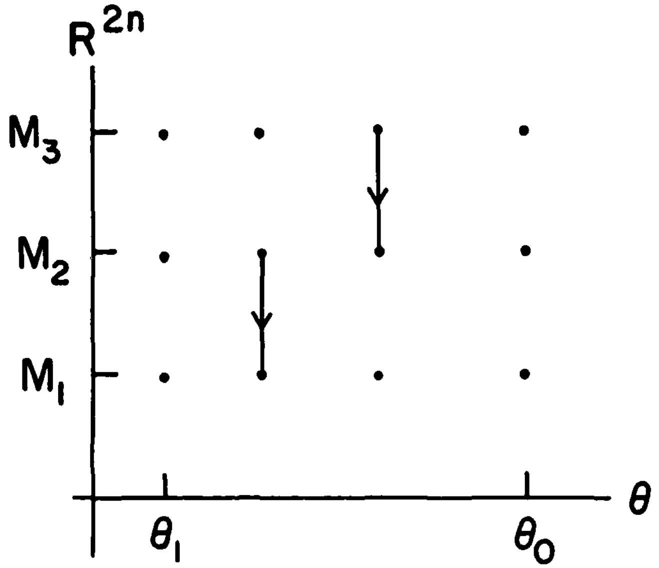
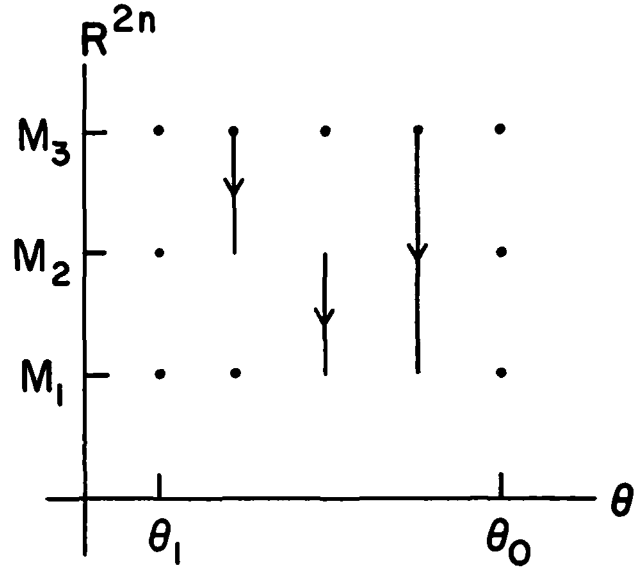

R^{2n}
M_i
M_j
θ
θ_1 θ^* θ_0
$\epsilon = 0$
R^{2n}
M_i $\bar{M}_i$ M_i
M_j $\bar{M}_j$ M_j
θ
θ_1 θ^* θ_0
$0 < \epsilon << 1$

FIG. 6

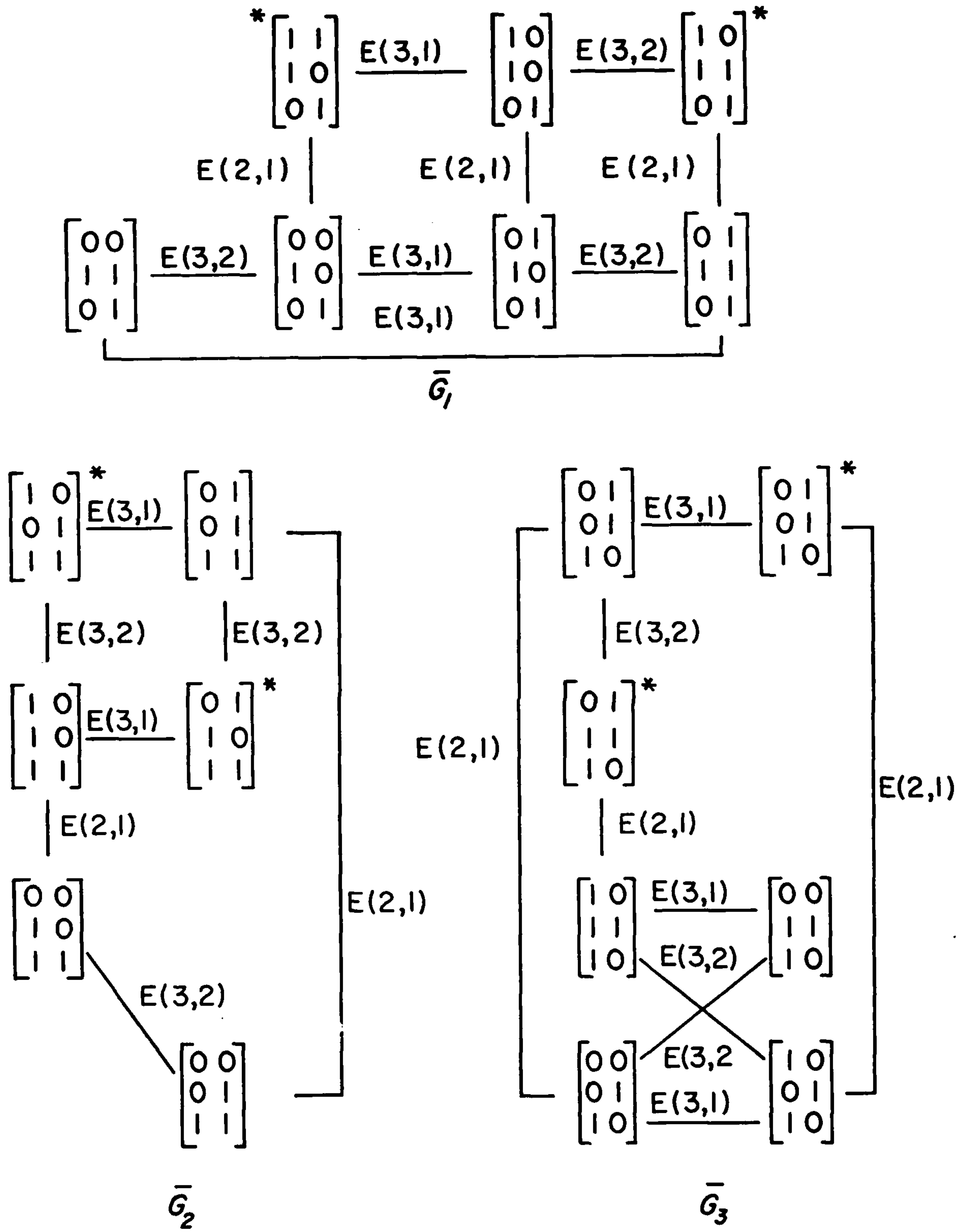


E(2,1) ∘ E(3,2)



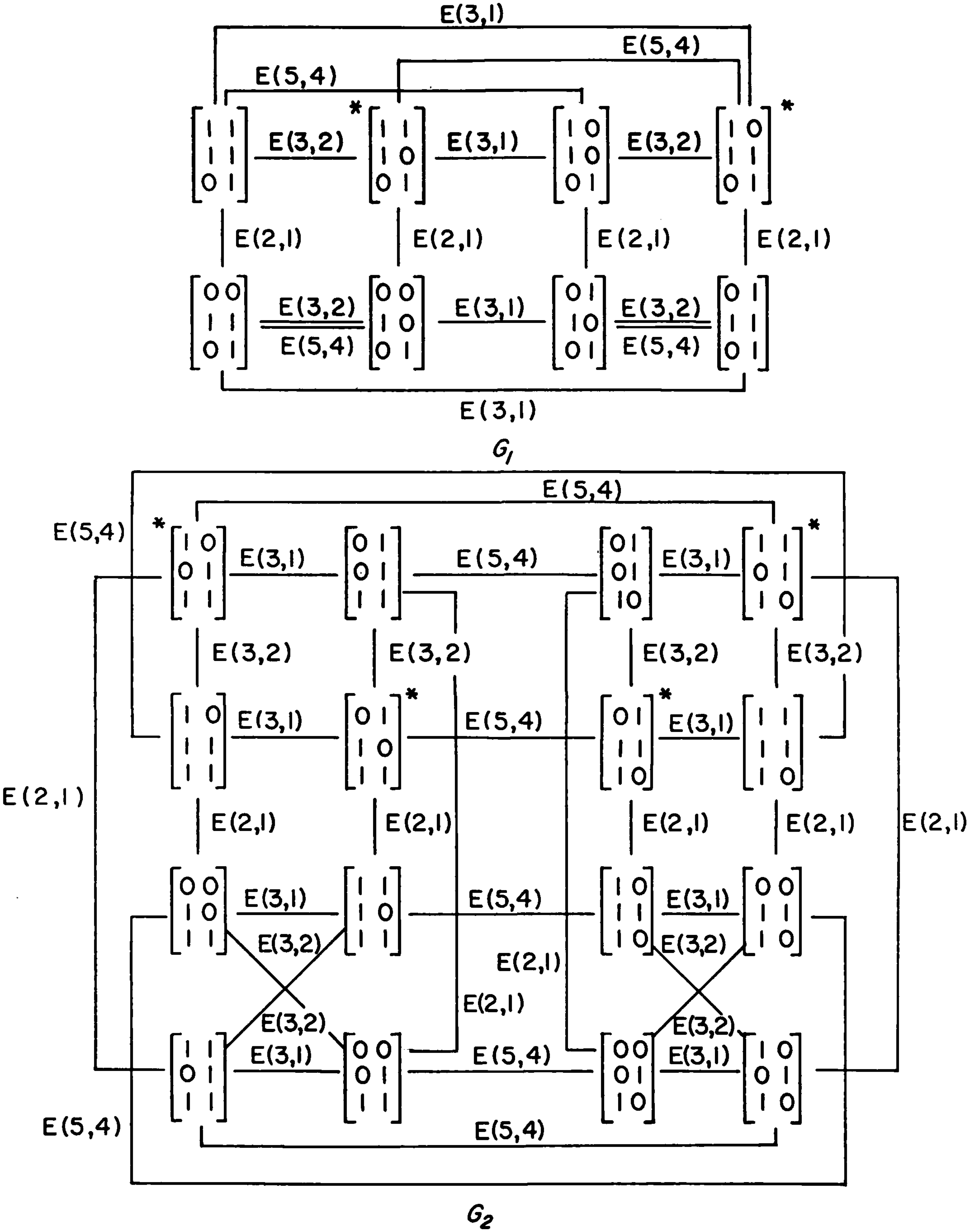
E(3,2) ∘ E(2,1) ∘ E(3,1)

FIG. 7



*denotes an element of $CM_L(\infty)$

FIG. 8



* denotes an element of $CM(\infty)$

FIG. 9

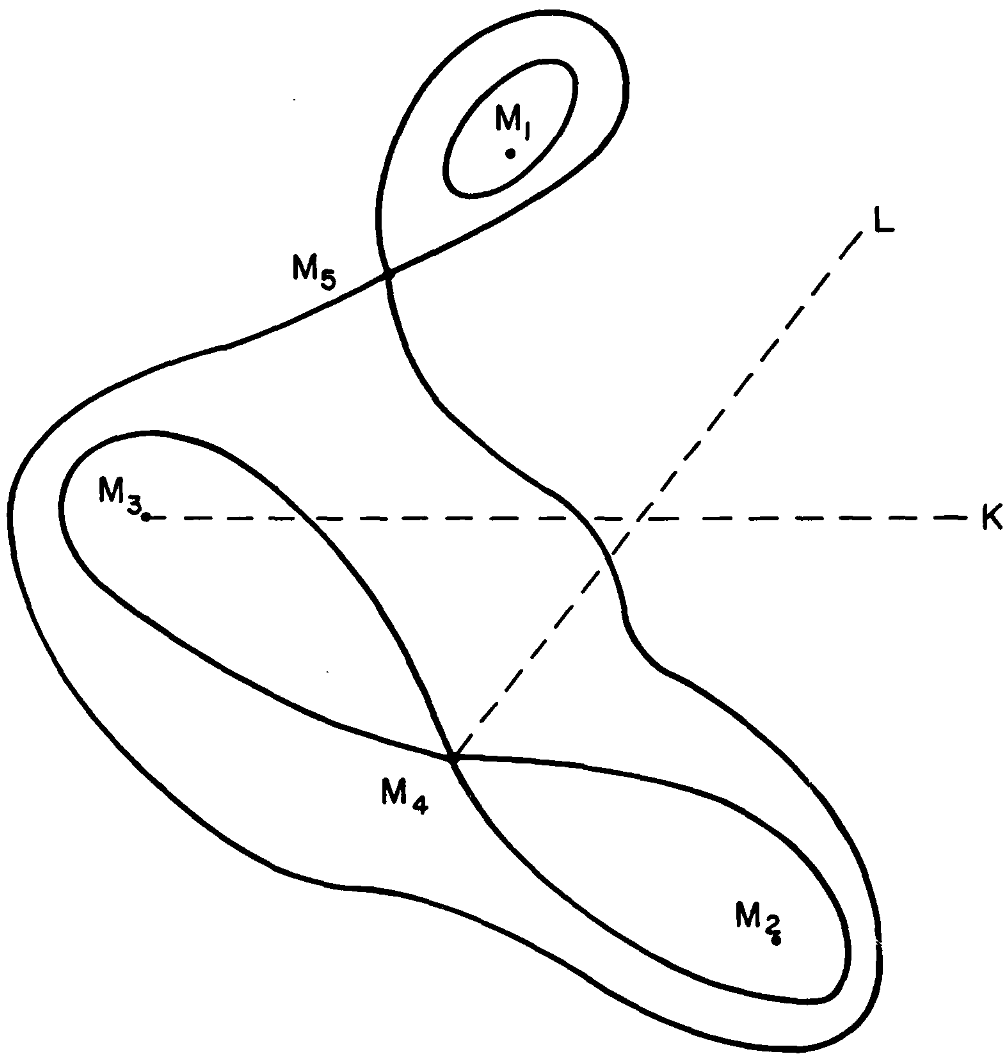


FIG. 10

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